Rendiconti di Matematica, Serie VII
Volume 28, Roma (2008), 19–32

Some highlights from the theory of multivariate symmetries

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Abstract: *We explain how invariance in distribution under separate or joint contractions, permutations, or rotations can be defined in a natural way for d-dimensional arrays of random variables. In each case, the distribution is characterized by a general representation formula, often easy to state but surprisingly complicated to prove. Comparing the representations in the first two cases, one sees that an array on a tetrahedral index set is contractable iff it admits an extension to a jointly exchangeable array on the full rectangular index set.*

Multivariate rotatability is defined most naturally for continuous linear random functionals on tensor products of Hilbert spaces. Here the simplest examples are the multiple Wiener–Itô integrals, which also form the basic building blocks of the general representations. The rotatable theory can be used to derive similar representations for separately or jointly exchangeable or contractable random sheets. The present paper provides a non-technical survey of the mentioned results, the complete proofs being available elsewhere. We conclude with a list of open problems.

1 – Basic symmetries and classical results

Many basic ideas in the area of probabilistic symmetries can be traced back to the pioneering work of Bruno de Finetti. After establishing, in 1930–37, his celebrated representation theorem for exchangeable sequences, he proposed in de Finetti (1938) the study of *partial exchangeability* of a random sequence, in the sense of invariance in distribution under a proper subgroup of permutations

Key Words and Phrases: *Contractable – Exchangeable and rotatable random arrays – Functionals and sheets – Multiple Wiener-Itô integrals.*
A.M.S. Classification: 60G09

of the elements. (A permutation on an infinite set is defined as a bijective map.) For a basic example, we may arrange the elements in a doubly infinite array and require invariance in distribution under permutations of all rows and all columns. This leads to the notion of *row-column* or *separate exchangeability*, considered below.

Many other probabilistic symmetries of interest can be described in terms of higher-dimensional arrays, processes, measures, or functionals. Their study leads to an extensive theory, whose current state is summarized in the last three chapters of the monograph KALLENBERG (2005)[(1)]. Our present aim is to give an informal introduction to some basic notions and results in the area. No novelty is claimed, apart from some open problems listed at the end of the paper. Before introducing the multivariate symmetries, we need to consider the one-dimensional case. For infinite sequences $X = (X_j)$ of random variables, we have the following basic symmetries, listed in the order of increasing strength. (Here X has the property on the left iff its distribution is invariant under the transformations on the right.)

stationary	*shifts*
contractable	*contractions*
exchangeable	*permutations*
rotatable	*rotations*

Thus, X is *contractable* if all subsequences have the same distribution, *exchangeable* if the joint distribution is invariant under arbitrary permutations, and *rotatable* if the distribution is invariant under any orthogonal transformation applied to finitely many elements. The notion of stationarity is well-known and will not be considered any further in this paper.

Sequences with the last three symmetry properties are characterized by the following classical results. Letting $X = (X_j)$ be an infinite sequence of random variables, we have:

- (DE FINETTI (1930, 1937)): *X is exchangeable iff it is mixed* (*or conditionally*) *i.i.d.*,
- (RYLL-NARDZEWSKI (1957)): *X is contractable iff it is exchangeable, hence mixed i.i.d.*,
- (FREEDMAN (1962)): *X is rotatable iff it is mixed i.i.d. centered Gaussian.*

For processes X on $\mathbb{R}_+$, the notions of contractability, exchangeability, and rotatability are defined in terms of the increments over any set of disjoint intervals of equal length. We may also assume that X is continuous in probability and starts at 0. Then the first two properties are again equivalent, and the three cases are characterized as follows:

[(1)] Henceforth abbreviated as K(2005).

- (Bühlmann (1960)): *X is exchangeable iff it is a mixture of Lévy processes,*
- (Freedman (1963)): *X is rotatable iff it is a mixture of centered Brownian motions with different rates.*

In particular, we see from the former result that a continuous process X is exchangeable iff it is a mixture of Brownian motions with arbitrary rate and drift coefficients. Thus, in the continuous case, the exchangeable and rotatable processes differ only by a random centering. This observation plays an important role for the analysis of exchangeable and contractable random sheets. Together with Ryll-Nardzewski's theorem, it also signifies a close relationship between the various symmetry properties in our basic hierarchy.

Proceeding to two-dimensional random arrays $X = (X_{ij};\, i,j \geq 1)$ indexed by $\mathbb{N}^2$, we may define the permuted arrays $X \circ (p,q)$ and $X \circ p$ by

$$(X \circ (p,q))_{ij} = X_{p_i,q_j}, \qquad (X \circ p)_{ij} = X_{p_i,p_j}\,,$$

where $p = (p_i)$ and $q = (q_j)$ are permutations on $\mathbb{N}$. Then X is said to be *separately exchangeable* if $X \circ (p,q) \stackrel{d}{=} X$ for all permutations p and q on $\mathbb{N}$ and *jointly exchangeable* if $X \circ p \stackrel{d}{=} X$ for any such permutation p. Note that the latter property is weaker, so that every separately exchangeable array is also jointly exchangeable. The definitions in higher dimensions are similar.

The contractable case is similar. Thus, X is said to be *separately contractable* if $X \circ (p,q) \stackrel{d}{=} X$ for all subsequences p and q of $\mathbb{N}$ and *jointly contractable* if $X \circ p \stackrel{d}{=} X$ for any such subsequence p. However, only the joint version is of interest, since the separate notions of exchangeability and contractability are equivalent by Ryll-Nardzewski's theorem above (applied to random elements in $\mathbb{R}^\infty$).

To define rotatability in higher dimensions, consider arrays $U = (U_{ij})$ such that, for some $n \in \mathbb{N}$, the restriction to the square $\{1,\ldots,n\}^2$ is orthogonal and otherwise $U_{ij} = \delta_{ij}$. For any such arrays U and V, we may define the array $X \circ (U \otimes V)$ by

$$(X \circ (U \otimes V))_{ij} = \sum\nolimits_{h,k} X_{hk}\, U_{hi}\, V_{kj}, \quad i,j \in \mathbb{N}\,,$$

and put $U^{\otimes 2} = U \otimes U$. Then X is said to be *separately rotatable* if $X \circ (U \otimes V) \stackrel{d}{=} X$ for all orthogonal arrays U and V as above and *jointly rotatable* if $X \circ U^{\otimes 2} \stackrel{d}{=} X$ for any such array U. Even these properties extend immediately to arbitrary dimensions.

The natural index set of a jointly exchangeable array X is not $\mathbb{N}^2$ but rather $\mathbb{N}^{(2)} = \{(i,j) \in \mathbb{N}^2;\, i \neq j\}$. In fact, an array X on $\mathbb{N}^2$ is jointly exchangeable iff the same property holds for the non-diagonal array

$$Y_{ij} = (X_{ij}, X_{ii}), \quad i \neq j\,.$$

(The latter property makes sense, since every permutation on $\mathbb{N}$ induces a joint permutation on $\mathbb{N}^{(2)}$.) Similarly, the natural index set for a d-dimensional, jointly exchangeable array is the set $\mathbb{N}^{(d)}$, consisting of all d-tuples $(i_1, \ldots, i_d)$ with distinct entries $i_1, \ldots, i_d$.

In the contractable case, we can go even further. Thus, an array X on $\mathbb{N}^2$ is jointly contractable iff the same property holds for the sub-diagonal (or super-diagonal, depending on the geometrical representation) array

$$Z_{ij} = (X_{ij}, X_{ji}, X_{ii}), \quad i < j \, .$$

Similarly, the natural index set for a d-dimensional, jointly contractable array is the *tetrahedral* index set

$$\Delta_d = \{(i_1, \ldots, i_d) \in \mathbb{N}^d; \, i_1 < \cdots < i_d\} \, .$$

It is often convenient to identify Δ_d with the class $\widetilde{\mathbb{N}}_d$, consisting of all subsets of $\mathbb{N}$ of cardinality d.

To summarize, we are led to consider exchangeable arrays on $\overline{\mathbb{N}} = \bigcup_d \mathbb{N}^{(d)}$ and contractable arrays on $\widetilde{\mathbb{N}} = \bigcup_d \widetilde{\mathbb{N}}_d$, where the qualification "jointly" is understood. The natural setting for the rotatable case will be discussed later.

2 – Exchangeable and contractable arrays

The aim of this section is to explain how separately or jointly exchangeable or contractable arrays of arbitrary dimension can be characterized by some general functional representations. In order to fully understand those formulas, it is useful to begin with the one-dimensional case. Write $U(0,1)$ for the uniform distribution on $[0,1]$.

- *An infinite random sequence $X = (X_j)$ is contractable (hence exchangeable) iff there exist a measurable function f on $[0,1]^2$ and some i.i.d. $U(0,1)$ random variables α and $\xi_1, \xi_2, \ldots$ such that a.s.*

$$X_j = f(\alpha, \xi_j), \quad j \geq 1 \, .$$

This is just another way of stating de Finetti's theorem. In particular, we see directly from this formula that the X_j are conditionally i.i.d. given α. This formulation has the disadvantage that the function f is not unique, and further that an independent randomization variable may be needed to construct the associated coding variables α and ξ_j.

For exchangeable arrays of higher dimension, the characterization problem is much harder. Here the first breakthrough came with Aldous' intricate proof (later simplified by Kingman) of the following result. (See also the elementary discussion in Aldous (1985).)

- (Aldous (1981)): *An array* $X = (X_{ij})$ *on* $\mathbb{N}^2$ *is separately exchangeable iff there exist a measurable function* f *on* $[0,1]^4$ *and some i.i.d.* $U(0,1)$ *random variables* α, ξ_i, η_j, *and* ζ_{ij} *such that a.s.*

$$X_{ij} = f(\alpha, \xi_i, \eta_j, \zeta_{ij}), \quad i, j \geq 1\,.$$

Representations of this type can also be deduced from certain results in formal logic, going back to the 1960's. (Here an elementary discussion appears in Hoover (1982).) Combining related methods with the techniques of non-standard analysis, Hoover found some general representations characterizing separately or jointly exchangeable arrays of arbitrary dimension. In the two-dimensional, jointly exchangeable case, his representation reduces to the following:

- (Hoover (1979)[2]): *An array* $X = (X_{ij})$ *on* $\mathbb{N}^{(2)}$ *is jointly exchangeable iff there exist a measurable function* f *on* $[0,1]^4$ *and some i.i.d.* $U(0,1)$ *random variables* α, ξ_i, *and* $\zeta_{\{i,j\}}$ *such that a.s.*

$$X_{ij} = f(\alpha, \xi_i, \xi_j, \zeta_{\{i,j\}}), \quad i \neq j\,.$$

Note that the representation in the separately exchangeable case follows as an easy corollary. Still deeper is the corresponding representation in the contractable case:

- (K (1992)): *An array* $X = (X_{ij})$ *on* Δ_2 *is jointly contractable iff there exist a measurable function* f *on* $[0,1]^4$ *and some i.i.d.* $U(0,1)$ *random variables* α, ξ_i, *and* ζ_{ij} *such that a.s.*

$$X_{ij} = f(\alpha, \xi_i, \xi_j, \zeta_{ij}), \quad i < j\,.$$

Comparing with the result in the jointly exchangeable case, we get the following rather surprising extension theorem:

- (K (1992)): *An array* $X = (X_{ij})$ *on* Δ_2 *is jointly contractable iff it can be extended to a jointly exchangeable array on* $\mathbb{N}^{(2)}$.

In fact, the last two results are clearly equivalent, given Hoover's representation in the jointly exchangeable case. No direct proof is known. As already mentioned, the representing function f in the quoted theorems is far from unique.

[2] The reason for the earlier date is that Hoover's long and difficult paper, written at the Institute of Advanced Study at Princeton, was never published.

To illustrate the possibilities, we may quote an equivalence criterion in the contractable case:

- (K (1992)): *Two measurable functions f and f' on $[0,1]^2$ can be used to represent the same contractable array X on Δ_2 iff there exist some measurable functions g_0, g_0' on $[0,1]$, g_1, g_1' on $[0,1]^2$, and g_2, g_2' on $[0,1]^4$, each measure preserving in the last argument, such that a.s., for any i.i.d. $U(0,1)$ random variables α, ξ_i, and ζ_{ij},*

$$\begin{aligned} &f(g_0(\alpha), g_1(\alpha,\xi_i), g_1(\alpha,\xi_j), g_2(\alpha,\xi_i,\xi_j,\zeta_{ij})) \\ &\quad = f'(g_0'(\alpha), g_1'(\alpha,\xi_i), g_1'(\alpha,\xi_j), g_2'(\alpha,\xi_i,\xi_j,\zeta_{ij})), \quad i<j\,. \end{aligned}$$

To state the higher-dimensional results in a concise form, we may introduce an array of i.i.d. $U(0,1)$ random variables (or *U-array*) $\xi = (\xi_J)$ indexed by $\widetilde{\mathbb{N}}$, and write

$$\hat{\xi}_J = (\xi_I;\, I \subset J), \quad J \in \widetilde{\mathbb{N}}\,.$$

Similarly, for any $k \in \overline{\mathbb{N}}$, we may form the associated set $\tilde{k} = \{k_1, k_2, \ldots\}$ and write

$$\hat{\xi}_k = (\xi_I;\, I \subset \tilde{k}), \quad k \in \overline{\mathbb{N}}\,.$$

To be precise, we also need to specify an order *among* (not *within*) the sets $I \subset \tilde{k}$, which is determined in an obvious way by the order within k of the elements k_j.

Using the previous terminology and notation and writing 2^n for the class of subsets of $\{1,\ldots,n\}$, we may state the general representations as follows:

- (Hoover (1979)): *An array X on $\overline{\mathbb{N}}$ is exchangeable iff there exist a measurable function f on $\bigcup_n [0,1]^{2^n}$ and a U-array ξ on $\widetilde{\mathbb{N}}$ such that a.s.*

$$X_k = f(\hat{\xi}_k), \quad k \in \overline{\mathbb{N}}\,.$$

- (K (1992)): *An array X on $\widetilde{\mathbb{N}}$ is contractable iff there exist a measurable function f on $\bigcup_n [0,1]^{2^n}$ and a U-array ξ on $\widetilde{\mathbb{N}}$ such that a.s.*

$$X_J = f(\hat{\xi}_J), \quad J \in \widetilde{\mathbb{N}}\,.$$

As before, the last result yields an associated extension theorem:

- (K (1992)): *An array X on $\widetilde{\mathbb{N}}$ is contractable iff it can be extended to an exchangeable array on $\overline{\mathbb{N}}$.*

3 – Rotatable arrays and functionals

Even separately or jointly rotatable arrays may be characterized in terms of a.s. representations. Here we may again begin with the one-dimensional case:

- (Freedman (1962)): *An infinite random sequence $X = (X_j)$ is rotatable iff there exist some i.i.d. $N(0,1)$ random variables $\zeta_1, \zeta_2, \ldots$ and an independent random variable $\sigma \geq 0$ such that a.s.*

$$X_j = \sigma \zeta_j, \quad j \geq 1 .$$

This is clearly equivalent to the previous characterization of rotatable sequences as mixed i.i.d. centered Gaussian. The two-dimensional case is again a lot harder. The following result, originally conjectured by Dawid (1978), was proved (under a moment condition) by an intricate argument based on the representation theorem for separately exchangeable arrays:

- (Aldous (1981)): *An array $X = (X_{ij})$ on $\mathbb{N}^2$ is separately rotatable iff there exist some i.i.d. $N(0,1)$ random variables ξ_{ki}, η_{kj}, and ζ_{ij}, along with an independent set of random coefficients σ and α_k satisfying $\sum_k \alpha_k^2 < \infty$, such that a.s.*

$$X_{ij} = \sigma \zeta_{ij} + \sum\nolimits_k \alpha_k \, \xi_{ki} \, \eta_{kj}, \quad i, j \geq 1 .$$

For jointly rotatable arrays, we have instead:

- (K (1988)): *An array $X = (X_{ij})$ on $\mathbb{N}^2$ is jointly rotatable iff there exist some i.i.d. $N(0,1)$ random variables ξ_{ki} and ζ_{ij}, along with an independent set of random coefficients ρ, σ, σ', and α_{hk} satisfying $\sum_{h,k} \alpha_{hk}^2 < \infty$, such that a.s.*

$$X_{ij} = \rho \delta_{ij} + \sigma \zeta_{ij} + \sigma' \zeta_{ji} + \sum\nolimits_{h,k} \alpha_{hk} \, (\xi_{hi} \, \xi_{kj} - \delta_{ij} \delta_{hk}), \quad i, j \geq 1 .$$

Here the centering terms $\delta_{ij} \delta_{hk}$ are needed, in general, to ensure convergence of the double series on the right.

The higher-dimensional representations are stated most conveniently in a Hilbert space setting. Here we consider any real, separable, infinite-dimensional Hilbert space H. By a *continuous linear random functional* (*CLRF*) on H we mean a real-valued process X on H such that

- $h_n \to 0$ in H implies $X h_n \xrightarrow{P} 0$,
- $X(ah + bk) = aXh + bXk$ a.s. for all $h, k \in H$ and $a, b \in \mathbb{R}$.

For a simple example, we may consider an *isonormal Gaussian process* (*G-process*) on H, defined as a centered Gaussian process X on H such that

$$\mathrm{Cov}(Xh, Xk) = \langle h, k \rangle, \quad h, k \in H .$$

By a *unitary operator* on H we mean a linear isometry U of H onto itself. A CLRF X on H is said to be *rotatable* if $X \circ U \stackrel{d}{=} X$ for all unitary operators U on H, where $(X \circ U)h = X(Uh)$. Taking $H = l^2$, we get the following equivalent version of Freedman's theorem:

- (Freedman (1962–63)): *A CLRF X on H is rotatable iff $X = \sigma\eta$ a.s. for some G-process η on H and an independent random variable $\sigma \geq 0$.*

This formulation has the advantage of also containing the corresponding continuous–time representation mentioned earlier–the fact that a continuous process X on $\mathbb{R}_+$ with $X_0 = 0$ is rotatable iff $X = \sigma B$ a.s. for some Brownian motion B and an independent random variable $\sigma \geq 0$. This amounts to choosing $H = L^2(\lambda)$, where λ denotes Lebesgue measure on $\mathbb{R}_+$.

The higher-dimensional representations are stated, most conveniently, in terms of rotations on tensor products of Hilbert spaces H_k. The latter are best understood when $H_k = L^2(\mu_k)$ for some σ-finite measures $\mu_1, \ldots, \mu_n$ on measurable spaces $S_1, \ldots, S_n$. The *tensor product* $\bigotimes_k H_k = H_1 \otimes \cdots \otimes H_n$ of the spaces H_k can then be defined by

$$H_1 \otimes \cdots \otimes H_n = L^2(\mu_1 \otimes \cdots \otimes \mu_n)\,,$$

where $\mu_1 \otimes \cdots \otimes \mu_n$ denotes the product measure of $\mu_1, \ldots, \mu_n$ on $S_1 \times \cdots \times S_n$. For any elements $h_k \in H_k$, we define the tensor product $\bigotimes_k h_k = h_1 \otimes \cdots \otimes h_n$ in $\bigotimes_k H_k$ by

$$(h_1 \otimes \cdots \otimes h_n)(s_1, \ldots, s_n) = h_1(s_1) \cdots h_n(s_n)\,,$$

for any $s_k \in S_k$, $k = 1, \ldots, n$. Choosing an orthonormal basis (ONB) $h_{k1}, h_{k2}, \ldots$ in H_k for every k, we note that the tensor products $\bigotimes_k h_{k,j_k}$ for arbitrary $j_1, \ldots, j_n \in \mathbb{N}$ form an ONB in $\bigotimes_k H_k$.

Given any unitary operators U_k on H_k, $k = 1, \ldots, n$, there exists a unique unitary operator $\bigotimes_k U_k = U_1 \otimes \cdots \otimes U_n$ on $\bigotimes_k H_k$ such that, for any elements $h_k \in H_k$,

$$(U_1 \otimes \cdots \otimes U_n)(h_1 \otimes \cdots \otimes h_n) = U_1 h_1 \otimes \cdots \otimes U_n h_n\,.$$

When $H_k = H$ or $U_k = U$ for all k, we may write $H^{\otimes n} = \bigotimes_k H_k$ or $U^{\otimes n} = \bigotimes_k U_k$, respectively. A CLRF X on $H^{\otimes n}$ is said to be *separately rotatable* if $X \circ \bigotimes_k U_k \stackrel{d}{=} X$ for all unitary operators $U_1, \ldots, U_n$ on H and *jointly rotatable* if $X \circ U^{\otimes n} \stackrel{d}{=} X$ for any such operator U. Basic examples are the *multiple Wiener–Itô integrals* (*WI-integrals*), defined most easily, as in K (2002), through the following characterizations (as opposed to the traditional lengthy constructions):

- *For any independent G-processes η_k on H_k, $k = 1, \ldots, n$, there exists an a.s. unique CLRF $\bigotimes_k \eta_k$ on $\bigotimes_k H_k$ such that, a.s. for any elements $h_k \in H_k$,*

$$(\eta_1 \otimes \cdots \otimes \eta_n)(h_1 \otimes \cdots \otimes h_n) = \eta_1 h_1 \cdots \eta_n h_n\,.$$

- *For any G-process η on H and any $n \in \mathbb{N}$, there exists an a.s. unique CLRF $\eta^{\otimes n}$ on $H^{\otimes n}$ such that, a.s. for any orthogonal elements $h_k \in H_k$,*

$$\eta^{\otimes n}(h_1 \otimes \cdots \otimes h_n) = \eta h_1 \cdots \eta h_n \,.$$

Similarly, we may define the multiple integral $\bigotimes_k \eta_k^{\otimes r_k}$ as a CLRF on $\bigotimes_k H_k^{\otimes r_k}$ for any $r_1, \ldots, r_n \in \mathbb{N}$. It is easily seen that $\bigotimes_k \eta_k$ is separately rotatable while $\eta^{\otimes n}$ is jointly rotatable. Note that the product formula for $\eta^{\otimes n}$ fails when the elements $h_1, \ldots, h_n$ are not orthogonal. In particular, we have the a.s. representation (due to Itô (1951))

$$\eta^{\otimes n} h^{\otimes n} = \|h\|^n p_n(\eta h/\|h\|), \quad h \in H,\ n \in \mathbb{N}\,,$$

where p_n denotes the n-th degree Hermite polynomial with leading coefficient 1.

To state the representation of separately rotatable random functionals, let $\mathcal{P}_d$ denote the set of partitions π of $\{1, \ldots, d\}$, and write $H^{\otimes J} = \bigotimes_{j \in J} H$ and $H^{\otimes \pi} = \bigotimes_{J \in \pi} H$.

- (K (1995)): *A CLRF X on $H^{\otimes d}$ is separately rotatable iff there exist some independent G-processes η_J on $H \otimes H^{\otimes J}$, $J \in 2^d \setminus \{\emptyset\}$, and an independent set of random elements $\alpha_\pi \in H^{\otimes \pi}$, $\pi \in \mathcal{P}_d$, such that a.s.*

$$Xf = \sum_{\pi \in \mathcal{P}_d} \left(\bigotimes_{J \in \pi} \eta_J \right) (\alpha_\pi \otimes f), \quad f \in H^{\otimes d}\,.$$

The last formula exhibits X as a finite sum of randomized multiple WI-integrals. Introducing an ONB $h_1, h_2, \ldots$ in H and writing

$$X_{k_1,\ldots,k_d} = X(h_{k_1} \otimes \cdots \otimes h_{k_d}), \quad k_1, \ldots, k_d \in \mathbb{N}\,,$$

we may write the previous representation in coordinate form as

$$X_k = \sum_{\pi \in \mathcal{P}_d} \sum_{l \in \mathbb{N}^\pi} \alpha_l^\pi \prod_{J \in \pi} \eta_{k_J, l_J}^J, \quad k \in \mathbb{N}^d\,,$$

for some i.i.d. $N(0,1)$ random variables η_{kl}^J and an independent collection of random elements α_l^π satisfying $\sum_l (\alpha_l^\pi)^2 < \infty$ a.s. Any separately rotatable array on $\mathbb{N}^d$ can be represented in this form. Note that in the functional version, the coefficients α_l^π have been combined into random elements α_π of H, which explains the role of the extra dimension of the G-processes η_J.

We turn to the more complicated jointly rotatable case. Here we write $\mathcal{O}_d$ for the class of partitions of $\{1, \ldots, d\}$ into *ordered* subsets $k = (k_1, \ldots, k_r) \in \mathbb{N}^{(r)}$, $1 \le r \le d$. The dimension r of k is denoted by $|k|$.

- (K (1995)): *A CLRF X on $H^{\otimes d}$ is jointly rotatable iff there exist some independent G-processes η_r on $H^{\otimes(r+1)}$, $r = 1, \dots, d$, and an independent set of random elements $\alpha_\pi \in H^{\otimes\pi}$, $\pi \in \mathcal{O}_d$, such that a.s.*

$$Xf = \sum_{\pi\in\mathcal{O}_d} \left(\bigotimes_{k\in\pi} \eta_{|k|}\right)(\alpha_\pi \otimes f), \quad f \in H^{\otimes d}.$$

Here the multiple integral $\bigotimes_{k\in\pi} \eta_{|k|}$ is understood to depend, in an obvious way, on the order of the elements within each sequence k.

The displayed formula may again be stated in basis form, using the mentioned expression of WI-integrals in terms of Hermite polynomials. However, the representation of jointly rotatable arrays is more complicated, as it also includes diagonal terms of different order. For example, the term $\rho\delta_{ij}$ in the quoted representation on $\mathbb{N}^2$ has no extension to a CLRF on $H^{\otimes 2}$. This shows another advantage of the Hilbert space setting, apart from the avoidance of infinite series involving Hermite polynomials.

4 – Exchangeable random sheets

We have already noted the close relationship between exchangeability and rotatability for continuous processes on $\mathbb{R}_+$. Exploring this connection, we may derive representations of certain separately or jointly exchangeable or contractable processes on $\mathbb{R}_+^d$ (and occasionally on $[0,1]^d$). By a *random sheet* on $\mathbb{R}_+^d$ we mean a continuous process $X = (X_t)$ that vanishes on all coordinate hyperplanes, so that $X_t = 0$ when $\bigwedge_j t_j = 0$. Note that exchangeability and rotatability may now be defined in an obvious way in terms of the multivariate increments.

To understand the higher-dimensional formulas, we may first consider the case of separately or jointly rotatable random sheets on $\mathbb{R}_+^2$. The following representations follow easily from the previous results for rotatable arrays.

- *A random sheet X on $\mathbb{R}_+^2$ is separately rotatable iff there exist some independent Brownian motions $B^1, B^2, \dots$ and $C^1, C^2, \dots$ and an independent Brownian sheet Z, along with an independent set of random coefficients σ and α_k with $\sum_k \alpha_k^2 < \infty$ a.s., such that a.s.*

$$X_{s,t} = \sigma Z_{s,t} + \sum\nolimits_k \alpha_k B_s^k C_t^k, \quad s, t \geq 0.$$

- *A random sheet X on $\mathbb{R}_+^2$ is jointly rotatable iff there exist some independent Brownian motions $B^1, B^2, \dots$ and an independent Brownian sheet Z,* *along with an independent set of random coefficients ρ, σ, σ', and α_k with $\sum_k \alpha_k^2 < \infty$ a.s., such that a.s.*

$$\begin{aligned} X_{s,t} = \rho(s \wedge t) + \sigma Z_{s,t} + \sigma' Z_{t,s} + \\ + \sum\nolimits_{h,k} \alpha_{hk} \, (B_s^h \, B_t^k - \delta_{hk}(s \wedge t)), \quad s,t \geq 0\,. \end{aligned}$$

The representations in the exchangeable case are similar, apart from some additional centering terms.

- (K (1988)): *A random sheet X on $\mathbb{R}_+^2$ is separately exchangeable iff there exist some independent Brownian motions B^k and C^k and an independent Brownian sheet Z, along with an independent set of random coefficients ϑ, σ, and $\alpha_k, \beta_k, \gamma_k$ with $\sum_k (\alpha_k^2 + \beta_k^2 + \gamma_k^2) < \infty$ a.s., such that a.s.*

$$X_{s,t} = \vartheta st + \sigma Z_{s,t} + \sum\nolimits_k (\alpha_k \, B_s^k \, C_t^k + \beta_k \, t B_s^k + \gamma_k \, s C_t^k), \quad s,t \geq 0\,.$$

- (K (1988)): *A random sheet X on $\mathbb{R}_+^2$ is jointly exchangeable iff there exist some independent Brownian motions B^k and an independent Brownian sheet Z, along with an independent set of random coefficients ρ, ϑ, σ, σ', and $\alpha_k, \beta_k, \beta'_k, \gamma_k$ with $\sum_k (\alpha_k^2 + \beta_k^2 + {\beta'}_k^2 + \gamma_k^2) < \infty$ a.s., such that a.s.*

$$\begin{aligned} X_{s,t} = \rho(s \wedge t) + \vartheta st + \sigma Z_{s,t} + \sigma' Z_{t,s} + \\ + \sum\nolimits_{h,k} \alpha_{hk} \, (B_s^h \, B_t^k - \delta_{hk}(s \wedge t)) + \\ + \sum\nolimits_k (\beta_k \, t B_s^k + \beta'_k \, s B_t^k + \gamma_k \, B_{s \wedge t}^k), \quad s,t \geq 0\,. \end{aligned}$$

Partial results of this type were also obtained, independently, in an unpublished thesis of Hestir (1986).

The higher-dimensional representations may again be stated, most conveniently, in terms of multiple WI-integrals. Here we write $\hat{\mathcal{P}}_d = \bigcup_I \mathcal{P}_I$, where $\mathcal{P}_I$ denotes the class of partitions π of $I \in 2^d \setminus \{\emptyset\}$. For $\pi \in \mathcal{P}_I$, we write $\pi^c = I^c$. Let λ^I denote Lebesgue measure on $\mathbb{R}_+^I$. For notational convenience, we may identify a set A with its indicator function 1_A.

- (K (1995)): *A random sheet X on $\mathbb{R}_+^d$ is separately exchangeable iff there exist some independent G-processes η_I on $H \otimes L^2(\lambda^I)$, $I \in 2^d \setminus \{\emptyset\}$, along with an independent set of random coefficients $\alpha_\pi \in H^{\otimes \pi}$, $\pi \in \hat{\mathcal{P}}_d$, such that a.s.*

$$X_t = \sum_{\pi \in \hat{\mathcal{P}}_d} \Big(\lambda^{\pi^c} \otimes \bigotimes_{I \in \pi} \eta_I \Big) (\alpha_\pi \otimes [0,t]), \quad t \in \mathbb{R}_+^d\,.$$

A similar representation holds for separately exchangeable random sheets on $[0,1]^d$, except that the G-processes η_J then need to be replaced by suitably "tied-down" versions.

The jointly exchangeable case is even more complicated and requires some further notation. Given a finite set J, we define $\hat{\mathcal{O}}_J = \bigcup_{I \subset J} \mathcal{O}_I$, where $\mathcal{O}_I$ denotes the class of partitions of I into ordered subsets k of size $|k| \geq 1$. In this definition, we may take J to be an arbitrary partition $\pi \in \mathcal{P}_d$, regarded as a finite collection of sets $\{J_1, \ldots, J_r\}$. For any $t \in \mathbb{R}_+^d$ and $\pi \in \mathcal{P}_d$, we introduce the vector $\hat{t}_\pi \in \mathbb{R}_+^\pi$ with components $\hat{t}_{\pi,J} = \bigwedge_{j \in J} t_j$, $J \in \pi$.

- (K (1995)): *A random sheet X on $\mathbb{R}_+^d$ is jointly exchangeable iff there exist some independent G-processes η_r on $H \otimes L^2(\lambda^r)$, $1 \leq r \leq d$, along with an independent set of random coefficients $\alpha_{\pi,\kappa} \in H^{\otimes\kappa}$, $\kappa \in \hat{\mathcal{O}}_\pi$, such that a.s.*

$$X_t = \sum_{\pi \in \mathcal{P}_d} \sum_{\kappa \in \hat{\mathcal{O}}_\pi} \left(\lambda^{\kappa^c} \otimes \bigotimes_{k \in \kappa} \eta_{|k|} \right) (\alpha_{\pi,\kappa} \otimes [0, \hat{t}_\pi]), \quad t \in \mathbb{R}_+^d .$$

One would expect the last representation to remain valid for jointly exchangeable sheets on $[0,1]^d$, with the G-processes η_r replaced by their tied-down versions. However, the status of this conjecture is still open. We may also mention some similar but still more complicated representations, available for jointly contractable sheets on $\mathbb{R}_+^d$ (cf. K (2005), p. 398). Finally, there exists an extensive theory of exchangeable random measures in the plane, covered by K (1990, 2005) but not included in the present survey.

In summary, the previous representations illustrate the amazing unity of the subject: using representations of contractable or exchangeable arrays, we may derive representations of rotatable random functionals in terms of multiple WI-integrals, which can then be used to obtain representations of exchangeable or contractable random sheets.

5 – Some open problems

The theory of multivariate symmetries is still incomplete. We conclude with a list of open problems in the area.

- Give a direct proof of the extension theorem for contractable arrays. This would provide an alternative approach to the deep representation theorem for such arrays, given Hoover's representation in the jointly exchangeable case. Some difficulties are likely to arise from the non-uniqueness, the fact that different representations may lead to different extensions. Is there a natural choice?
- Characterize the jointly exchangeable random sheets on $[0,1]^d$. One expects the representation on $\mathbb{R}_+^d$ to remains valid with the G-processes η_r replaced

by their tied-down versions. Hence, the question is whether there exist exchangeable sheets that are not given by this formula.

- Find representations of arrays and processes with different symmetries (contractable, exchangeable, or rotatable) in different indices or variables.
- Characterize the classes of separately or jointly exchangeable random measures on $\mathbb{R}_+^d$ for $d \geq 3$. The complexity of such representations, already for $d = 2$ (cf. K (1990, 2005)), suggests that one should first look for a compact way of writing these formulas, starting perhaps with the special case of simple point processes.
- Extend Bühlmann's (1960) theorem to higher dimensions, by characterizing processes on $\mathbb{R}_+^d$ with separately or jointly exchangeable increments. It seems reasonable to begin with the case of signed random measures on $\mathbb{R}_+^2$.
- Multiple Wiener-Itô integrals constitute the basic examples of rotatable arrays and functionals in higher dimensions. Are there any natural symmetries leading to multiple p-stable integrals for $p < 2$? For the one-dimensional case, see e.g. Diaconis and Freedman (1987).
- Can the semigroup methods of Ressel (1985) be used to derive the representations of separately or jointly rotatable arrays and functionals? In view of the complexity of the current proofs, it seems worthwhile looking for alternative approaches.

Acknowledgements

My sincere thanks to the Editors for providing a long list of helpful comments and suggestions.

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Lavoro pervenuto alla redazione il 12 ottobre 2007
ed accettato per la pubblicazione il 18 ottobre 2007.
Bozze licenziate il 9 aprile 2008

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