Stochastic flows associated to coalescent processes

Jean Bertoin[1] and Jean-François Le Gall[2]

(1) *Laboratoire de Probabilités et Modèles Aléatoires and Institut universitaire de France, Université Pierre et Marie Curie, 175, rue du Chevaleret, F-75013 Paris, France.*

(2) *DMA, Ecole Normale Supérieure, 45, rue d'Ulm, F-75005 Paris, France.*

Summary. We study a class of stochastic flows connected to the coalescent processes that have been studied recently by Möhle, Pitman, Sagitov and Schweinsberg in connection with asymptotic models for the genealogy of populations with a large fixed size. We define a bridge to be a right-continuous process $(B(r), r \in [0,1])$ with nondecreasing paths and exchangeable increments, such that $B(0) = 0$ and $B(1) = 1$. We show that flows of bridges are in one-to-one correspondence with the so-called exchangeable coalescents. This yields an infinite-dimensional version of the classical Kingman representation for exchangeable partitions of $\mathbb{N}$. We then propose a Poissonian construction of a general class of flows of bridges and identify the associated coalescents. We also discuss an important auxiliary measure-valued process, which is closely related to the genealogical structure coded by the coalescent and can be viewed as a generalized Fleming-Viot process.

1   Introduction

The purpose of this work is to investigate a class of stochastic flows which are naturally connected to a family of processes of coalescence. The latter appear as models for the genealogy of populations when the size of the population tends to infinity, and have been introduced recently by Möhle, Pitman, Sagitov and Schweinsberg. To explain this as simply as possible, let us start by discussing a discrete elementary setting.

We consider a haploid population model with a fixed size N and non-overlapping generations. Thus, we may and will identify the population at each generation with $\{1, \ldots, N\}$. For every generation $n \in \mathbb{Z}$ and $1 \le i \le N$, we denote by $\xi_{i,n}$ the number of children of the individual i at generation n. Since the total size N of the population is fixed, we must have $\xi_{1,n} + \cdots + \xi_{N,n} = N$.

We agree to assign labels to individuals in a way which is compatible with the genealogy, in the sense that when $\xi_{i,n} \geq 1$, the labels at the generation $n+1$ of the children of the individual i at the n-th generation run from $\xi_{1,n} + \cdots + \xi_{i-1,n} + 1$ to $\xi_{1,n} + \cdots + \xi_{i,n}$. We may thus represent the offspring of the n-th generation by the function $\Delta_n : \{0, \ldots, N\} \to \{0, \ldots, N\}$ which is defined by

$$\Delta_n(0) = 0 \text{ and } \Delta_n(i) = \xi_{1,n} + \cdots + \xi_{i,n} \text{ for } i = 1, \ldots, N.$$

By iteration, the total number of descendants at the generation $n+k$ of the individual i at the generation n is given by the i-th increment of the compound function $\Delta_{n+k-1} \circ \cdots \circ \Delta_n$, that is $\Delta_{n+k-1} \circ \cdots \circ \Delta_n(i) - \Delta_{n+k-1} \circ \cdots \circ \Delta_n(i-1)$.

We now suppose that the preceding data are random as in the model of Cannings [5, 6]. More precisely, for each $n \in \mathbb{Z}$, the N-tuple $(\xi_{1,n}, \ldots, \xi_{N,n})$ is exchangeable, i.e. its distribution is invariant under any permutation of indices, and the sequence $\{(\xi_{1,n}, \ldots, \xi_{N,n}), n \in \mathbb{Z}\}$ is i.i.d. In other words, each map Δ_n is a discrete bridge from 0 to N with exchangeable increments, and the sequence of maps $\{\Delta_n, n \in \mathbb{Z}\}$ is i.i.d. We then set for every $m, n \in \mathbb{Z}$, with $m < n$

$$B_{m,n} := \Delta_{-m+1} \circ \cdots \circ \Delta_{-n}.$$

The reason for this particular definition is easy to understand if we think of tracing back the genealogical history of individuals: For $n \geq 1$, the increments of $B_{0,n}$ are exactly the sizes of the blocks of the partition of $\{1, \ldots, N\}$ corresponding to individuals at generation 0 who have the same ancestor n generations backwards in time. We also set $B_{m,m} = \text{Id}$ and note that we have the following two properties:

(i) (*cocycle property*) For every $\ell \leq m \leq n$

$$B_{\ell,m} \circ B_{m,n} = B_{\ell,n}. \tag{1}$$

(ii) The law of $B_{m,n}$ only depends of $n-m$, and the bridges $B_{n_1,n_2}, \ldots, B_{n_{k-1},n_k}$ are independent for every $n_1 \leq \cdots \leq n_k$.

We call $(B_{m,n} : m, n \in \mathbb{Z} \text{ and } m \leq n)$ a *flow* of discrete bridges (with exchangeable increments). The connection with coalescence is clear from the genealogical interpretation: For every $0 < m < n$, the increments of $B_{0,n}$ are obtained by a certain coagulation (i.e. addition) of those of $B_{0,m}$, and more precisely the coagulation mechanism is encoded by $B_{m,n}$ via the cocycle identity (1).

In this work, we shall be interested in studying the continuous version of the discrete model above. On one hand, the continuous analog of a discrete bridge is a càdlàg random process $B = (B(r), r \in [0,1])$ with nondecreasing paths and exchangeable increments, such that $B(0) = 0$ and $B(1) = 1$. These processes will simply be called *bridges* in the next sections. They have been studied in detail by Kallenberg [10]. A *flow of bridges* is then a collection $(B_{s,t}, -\infty < s \leq t < \infty)$ of bridges, which satisfies the obvious analogs of properties (i) and (ii) (we also require a mild continuity assumption, see Section 3 below).

On the other hand, recent papers by Möhle, Sagitov, Pitman and Schweinsberg (cf. [14, 15, 17, 18]) have exhibited a large family of coalescent processes with values in the space $\mathcal{P}$ of partitions of $\mathbb{N} := \{1, 2, \ldots\}$, which arise as the limits of Cannings' model for genealogy when

the size N of the population tends to infinity. These processes are called here exchangeable $\mathcal{P}$-coalescents. To be specific, an exchangeable $\mathcal{P}$-coalescent is a Markov process $(\Pi_t, t \geq 0)$ with values in $\mathcal{P}$, with a Feller semigroup (P_t) satisfying the following property: For every $t > 0$ and $\pi \in \mathcal{P}$, $P_t(\pi, \cdot)$ is the law of the coagulation of π by a (random) exchangeable partition π_t depending only on t (if $\pi, \pi' \in \mathcal{P}$, and $A_1, A_2, \ldots$ are the blocks of π ranked in the increasing order of their smallest elements, the coagulation of π by π' is the new partition whose blocks are the sets $\cup_{i \in C} A_i$ when C varies over the blocks of π'). Exchangeable $\mathcal{P}$-coalescents exactly correspond to the class studied by Schweinsberg [18]. An important special case is the class of coalescents with multiple collisions (or Λ-coalescents) studied in [15].

Our first theorem (Theorem 1) shows that flows of bridges are in one-to-one correspondence with exchangeable $\mathcal{P}$-coalescents. One way to express this correspondence is to say that, for any flow of bridges B, the ranked sequence of jumps of $B_{0,t}$, viewed as a process in the variable t, has the same distribution as the ranked sequence of frequencies of an exchangeable $\mathcal{P}$-coalescent at time t. Note that a first result in the direction of Theorem 1 is given as Lemma 121 of Pitman [16].

In Section 4, we develop a Poissonian construction à la Lévy-Itô of flows of bridges. The idea there is to compose elementary bridges with exactly one jump, which are of the form

$$b_{U,x}(r) = (1 - x)r + x\,1_{\{U \leq r\}}, \quad r \in [0, 1]$$

where $x \in\,]0, 1]$ and the random variable U is uniform over the interval $[0, 1]$. Precisely, we let ν be a finite measure on $]0, 1]$ and denote by M a Poisson point measure on $\mathbb{R} \times]0, 1[\times]0, 1]$ with intensity $dt \otimes du \otimes \nu(dx)$. We then set for every $s < t$

$$B_{s,t}^M = b_{u_1,x_1} \circ \cdots \circ b_{u_K,x_K},$$

where $(t_1, u_1, x_1), \ldots, (t_K, u_K, x_K)$ are the atoms of M in $]s, t] \times]0, 1[\times]0, 1]$, ordered so that $s < t_1 < \cdots < t_K \leq t$. Then B^M is a flow of bridges. Flows of bridges that are constructed in this way are called *simple*. We prove in Section 4 that the flow of bridges associated with a general Λ-coalescent can be obtained as a weak limit of simple flows of bridges (Theorem 2).

In Section 5, we study the dual flow of a general flow of bridges B, which is defined by $\hat{B}_{s,t} = B_{-t,-s}$ for $-\infty < s \leq t < \infty$. We are especially interested in the process (ρ_t) taking values in the set $\mathcal{M}_1$ of all probability measures on $[0, 1]$, which is defined by $\rho_t([0, r]) = \hat{B}_{0,t}(r)$ for every $r \in [0, 1]$. The process (ρ_t) is a Markov process in $\mathcal{M}_1$ with a Feller semigroup. By analogy with the discrete case discussed previously, we can think of $\rho_t(dr)$ as the size of the progeny at time t of the fraction dr of the population at time 0 (in the continuous setting, the population is identified with the interval $[0, 1]$). When the flow B is simple, the process (ρ_t) is a continuous-time Markov chain with values in $\mathcal{M}_1$, which can be seen as a generalized Fleming-Viot process. Using an approximation by simple flows, we show in subsection 5.2 that in the case of the flow associated with a general Λ-coalescent, the law of (ρ_t) is characterized by a martingale problem for which uniqueness is obtained through a duality argument. To be specific, the process (ρ_t) appears as a measure-valued dual to the Λ-coalescent. This is of course reminiscent of the classical duality for Fleming-Viot processes (see e.g. Chapter 1 of [7]). In the last part of Section 5, we study the one-point motions $t \to \hat{B}_{0,t}(r) = \rho_t([0, r])$ for a fixed $r \in [0, 1]$. We are especially interested in the *primitive Eve* $\mathbf{e}$ of the population, which

is characterized by the property $\rho_t([\mathbf{e} - \varepsilon, \mathbf{e} + \varepsilon]) \to 1$ as $t \to \infty$, for every $\varepsilon > 0$. We give a description of the law of the process $(\rho_t(\{\mathbf{e}\}))_{t \geq 0}$ of the progeny of $\mathbf{e}$. More precisely, we show that $(\rho_t(\{\mathbf{e}\}))_{t \geq 0}$ is a Feller process and we specify its semigroup and its entrance distributions.

As suggested by the end of subsection 3.2, it is also of interest to study the several points motion for the flow of inverses $B_{s,t}^{-1}$. Under certain conditions, this leads to a coalescing flow on the unit interval, in the sense of Le Jan and Raimond [13]. This coalescing flow will be studied in the forthcoming paper [2].

2 Preliminaries

2.1 Bridges

A *bridge* is a random process $B = (B(r), r \in [0,1])$ with values in the interval $[0,1]$, such that

(i) $B(0) = 0$, $B(1) = 1$ and the paths of B are right-continuous and nondecreasing.

(ii) B has exchangeable increments.

According to the general results on processes with exchangeable increments [10], a process B is a bridge if and only if there exists a sequence of nonnegative random variables $(\beta^i, i = 1, 2, \ldots)$, with $\beta^1 \geq \beta^2 \geq \beta^3 \geq \cdots$, and $\sum_{i=1}^{\infty} \beta^i \leq 1$, and a sequence of i.i.d. uniform $[0,1]$ variables $(U^i, i = 1, 2, \ldots)$, also independent of the sequence (β^i), such that a.s. for every $r \in [0,1]$,

$$B(r) = \Big(1 - \sum_{i=1}^{\infty} \beta^i\Big) r + \sum_{i=1}^{\infty} \beta^i \, 1_{\{U^i \leq r\}}. \tag{2}$$

Remark. For the preceding representation to hold it may be necessary to enlarge the underlying probability space. What really matters is the fact that for any bridge B there is another bridge with the same distribution as B and such that the representation (2) holds.

Note that we may have $\beta^i = 0$ for i large. Still we will refer to the sequence (β^i) as the ranked sequence of jump sizes of B. Obviously the law of B is determined by that of the sequence (β^i).

Let $\mathcal{S}$ stand for the set of all nonincreasing sequences $x = (x_i, i = 1, 2, \ldots)$ such that $|x| := \sum_{i=1}^{\infty} x_i \leq 1$. The set $\mathcal{S}$ is equipped with the uniform distance

$$d(x, y) = \sup_i |x_i - y_i|.$$

Then $(\mathcal{S}, d)$ is a compact metric space. We let $\mathcal{S}_f$ be the subset of $\mathcal{S}$ defined by

$$\mathcal{S}_f = \{x \in \mathcal{S} : \sum_{i=1}^{k} x_i = 1 \text{ for some } k < \infty\}.$$

Note that $\mathcal{S}_f$ is dense in $\mathcal{S}$.

If B is a bridge, the ranked sequence of jump sizes of B is a random variable with values in $\mathcal{S}$. When dealing with weak convergence of random variables with values in $\mathcal{S}$, we will always refer to the topology induced by the distance d.

2.2 Exchangeable partitions

We write $\mathbb{N} = \{1, 2, \ldots\}$ for the set of positive integers. A *partition* π of $\mathbb{N}$ can be represented as a sequence $(A_1, A_2, \ldots)$ of disjoint subsets of $\mathbb{N}$ such that

$$\bigcup_{i \in \mathbb{N}} A_i = \mathbb{N}$$

and the blocks A_i are ranked according to the increasing order of their least element, i.e.

$$i \leq j \implies \min A_i \leq \min A_j \, ,$$

with the convention $\min \emptyset = \infty$. The partition has finitely many (nonempty) blocks if and only if $A_i = \emptyset$ for all i sufficiently large. We write $\mathcal{P}$ for the set of all partitions of $\mathbb{N}$ equipped with the usual distance, i.e. the distance between two distinct partitions π and π' belongs to $\{2^{-k}, k \in \mathbb{N}\}$ and this distance is less than 2^{-n} if and only if the restrictions of π and π' to $\{1, \ldots, n\}$ coincide. Then $\mathcal{P}$ is a compact metric space.

It is sometimes convenient to view a partition π as an equivalence relation, in the sense that $i \overset{\pi}{\sim} j$ if and only if i and j belong to the same block of the partition π.

Following Kingman [11], a random partition (i.e. a $\mathcal{P}$-valued random variable) is called *exchangeable* if its distribution is invariant under the action of permutations of $\mathbb{N}$. If Π is an exchangeable partition, the asymptotic frequency

$$f_A = \lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$$

exists for every block A of Π, a.s. (here we denote by $|F|$ the cardinality of a finite set F). The random element of $\mathcal{S}$ whose nonzero values are exactly the (nonzero) frequencies of the blocks of Π will be called the ranked sequence of frequencies of Π. The law of Π is completely determined by that of its ranked sequence of frequencies. Conversely, given a probability distribution on $\mathcal{S}$, there exists an exchangeable partition of $\mathbb{N}$ whose ranked sequence of frequencies has the given distribution. This exchangeable partition may be constructed by Kingman's paintbox scheme, a variant of which is presented in Section 2.3 below.

If π and π' are two (deterministic) partitions, we call the *coagulation* of π by π' and denote by $c_{\pi'}(\pi)$ the partition whose blocks are given by

$$\bigcup_{i \in A'} A_i$$

where $A_1, A_2, \ldots$ stand for the blocks of π and A' varies over the blocks of π'. (In [16], Definition 19, $c_{\pi'}(\pi)$ is called the π'-coagulation of π.) Note that the map $c_{\pi'} : \mathcal{P} \to \mathcal{P}$ is continuous for any partition $\pi' \in \mathcal{P}$.

2.3 Exchangeable partitions and bridges

Let $B = (B(r), r \in [0, 1])$ be a bridge. The (right-continuous) inverse B^{-1} of B is defined by the formula

$$B^{-1}(r) = \inf\{s \in [0, 1] : B(s) > r\} \text{ if } r \in [0, 1[, \ B^{-1}(1) = 1.$$

The ranked sequence of the lengths of the maximal intervals on which B^{-1} remains constant coincides with that of the jump sizes of B.

Let $V_1, V_2, \ldots$ be an infinite sequence of i.i.d. uniform random variables on $[0, 1]$, which is independent of B, and define a random partition $\pi(B)$ by the equivalence relation

$$i \overset{\pi(B)}{\sim} j \iff B^{-1}(V_i) = B^{-1}(V_j), \qquad i, j \in \mathbb{N}.$$

Note that this random partition depends on the sequence $V_1, \ldots$, although for convenience this is not indicated in the notation. In what follows we will consider different choices of the bridge B, but we will *always* use the same independent sequence (V_i) to define $\pi(B)$.

The partition $\pi(B)$ is exchangeable and the ranked sequence of its frequencies is the ranked sequence of jump sizes of B. Of course, the definition of $\pi(B)$ is merely a variation of Kingman's paintbox scheme (see e.g. Theorem 36 of [15]) which sets up a bijection between probability distributions for an exchangeable partition of $\mathbb{N}$ and probability distributions on $\mathcal{S}$.

We start by stating a continuity lemma which combines Kingman's continuity theorem (see e.g. Theorem 36 in [15]) and a result of Kallenberg for the weak convergence of bridges with exchangeable increments (see Theorem 2.3 in [10]).

Lemma 1 (Continuity Lemma) *Consider for each $n \in \mathbb{N} \cup \{\infty\}$ a bridge B_n and denote by $\beta_n = (\beta_n^1, \beta_n^2, \ldots)$ the ranked sequence of its jump sizes. The following conditions are equivalent:*

(i) *When $n \to \infty$, the exchangeable partition $\pi(B_n)$ converges in distribution to $\pi(B_\infty)$.*

(ii) *When $n \to \infty$, the sequence of ranked jump sizes β_n converges in distribution to β_∞ in $\mathcal{S}$.*

(iii) *When $n \to \infty$, the bridge B_n converges in distribution to B_∞ in the space $\mathbb{D}([0, 1], \mathbb{R}_+)$ endowed with the Skorohod topology.*

Remark. A little more generally, if we start from a sequence $(B_n)_{n \in \mathbb{N}}$ of bridges and assume that the exchangeable partitions $\pi(B_n)$ converge in distribution, then the limit has to be of the form $\pi(B_\infty)$ and properties (i),(ii) and (iii) hold.

Assume for a while that the ranked sequence $\beta^1 \geq \beta^2 \geq \cdots$ of the jump sizes of B is deterministic. We shall refer to the *discrete case* when this ranked sequence is in $\mathcal{S}_f$, i.e. $\sum_{i=1}^k \beta^i = 1$ for some large enough k, and to the *continuous case* otherwise. Plainly, the continuous case occurs iff B^{-1} has continuous paths a.s., whereas the bridge B is a step process in the discrete case. Also note that $\pi(B)$ has infinitely many blocks in the continuous case, a.s., and only finitely many blocks in the discrete case, a.s.

Let us come back to the case of a general bridge B. Write $\beta = (\beta^1, \ldots)$ for the ranked sequence of jumps of B and $A_1, A_2, \ldots$ for the (ordered) sequence of the blocks of $\pi(B)$. On the event $\{\beta \notin \mathcal{S}_f\}$, all blocks $A_i, i = 1, 2, \ldots$ are nonempty a.s., and thus we may define for every $i \in \mathbb{N}$

$$V_i' = B^{-1}(V_j) \qquad \text{for an arbitrary } j \in A_i. \tag{3}$$

On the event $\{\beta \in \mathcal{S}_f\}$, let K denote the (random) number of nonzero components of β. Then $\pi(B)$ has exactly K non-empty blocks, a.s., and we may define V_i' by (3), but only for $i = 1, \ldots, K$. It is then convenient to further introduce an i.i.d. sequence of uniform variables

$W_i, i \in \mathbb{N}$ which is independent of the preceding quantities, and to put $V_i' = W_i$ for $i > K$. These additional variables are only used to unify the presentation and will play essentially no role in the sequel.

Lemma 2 *Let B be a bridge. We have:*

(i) *The partition $\pi(B)$ and the sequence $V_1', \ldots$ are independent.*

(ii) *The variables $V_1', \ldots$ are i.i.d. and uniformly distributed on $[0, 1]$.*

Proof: By a simple conditioning argument, we may assume that the ranked sequence $\beta^1 \geq \beta^2 \geq \cdots$ of jumps of B is deterministic, and we let $k \leq +\infty$ be the number of nonzero components in this sequence.

We first consider the discrete case. Recall that U_i is the instant of the jump of B with size β^i, and that $U_1, \ldots, U_k$ are i.i.d. uniform variables on $[0, 1]$. Let $U_1^* < \cdots < U_k^*$ be the order statistics of $U_1, \ldots, U_k$, and denote by σ the permutation of $\{1, \ldots, k\}$ such that $U_i^* = U_{\sigma(i)}$. In particular, σ is independent of the U_i^*'s and is uniformly distributed.

Next, consider the random interval partition of $[0, 1[$ induced by the range of B,

$$I_1 = [0, B(U_1^*)[\quad , \quad I_2 = [B(U_1^*), B(U_2^*)[\quad , \quad \cdots \quad , \quad I_k = [B(U_{k-1}^*), 1[.$$

Note that the length of I_i is $\beta^{\sigma(i)}$, so the intervals $I_1, \ldots, I_k$ are independent of the order statistics $U_1^* < \cdots < U_k^*$ and are exactly the intervals on which B^{-1} remains constant (more precisely $B^{-1} \equiv U_i^*$ on I_i).

We define a second random permutation τ of $\{1, \ldots, k\}$ as follows: For $i = 1, \ldots, k$, $\tau(i)$ is the index of the i-th interval amongst $I_1, \ldots, I_k$ visited by the sequence $V_1, \ldots$, in the sense that $V_1 \in I_{\tau(1)}$, and if $V_1, \ldots, V_{j-1} \in I_{\tau(1)}$ and $V_j \notin I_{\tau(1)}$, then $\tau(2)$ is specified by $V_j \in I_{\tau(2)}$, and so on. From the very definition of the V_i''s and that of the blocks $A_1, \ldots, A_k$ of the partition $\pi(B)$, we see that for $i = 1, \ldots, k$,

$$A_i = \left\{ n : V_n \in I_{\tau(i)} \right\} \quad \text{and} \quad V_i' = U_{\tau(i)}^*. \tag{4}$$

On the one hand, because $V_1, \ldots$ is a sequence of i.i.d. uniform variables which is independent of the bridge B, the conditional distribution of $(\tau, \pi(B))$ given B only depends on the sequence of the lengths of the intervals $I_1, \ldots, I_k$, viz. $\beta^{\sigma(1)}, \ldots, \beta^{\sigma(k)}$. As a consequence,

$$(\tau, \pi(B)) \text{ is independent of the order statistics } U_1^* < \cdots < U_k^*. \tag{5}$$

On the other hand, we claim that

$$\tau \text{ is independent of the partition } \pi(B), \tag{6}$$

and

$$\tau \text{ is uniformly distributed on the set of permutations of } \{1, \ldots, k\}. \tag{7}$$

Indeed, fix an arbitrary permutation η of $\{1, \ldots, k\}$. Because $\sigma \circ \eta$ is again a random uniform permutation independent of the order statistics $U_1^* < \cdots < U_k^*$, the step process $\tilde{B}$ on the unit

interval which has a jump of size $\beta^{\sigma \circ \eta(i)}$ at time U_i^* for each $i = 1, \ldots, k$, is a bridge with the same law as B. Next, set

$$\tilde{I}_1 = [0, \tilde{B}(U_1^*)[\quad , \quad \tilde{I}_2 = [\tilde{B}(U_1^*), \tilde{B}(U_2^*)[\quad , \quad \ldots \quad , \quad \tilde{I}_k = [\tilde{B}(U_{k-1}^*), 1[,$$

so that $|\tilde{I}_i| = \beta^{\sigma \circ \eta(i)}$. There is a unique càdlàg bijection $\varphi : [0, 1[\to [0, 1[$ which is linear with unit slope on $\tilde{I}_i$ and such that $\varphi(\tilde{I}_i) = I_{\eta(i)}$ for $i = 1, \ldots, k$. Since the inverse φ^{-1} preserves the Lebesgue measure, if we set $\tilde{V}_n = \varphi^{-1}(V_n)$ for every $n \in \mathbb{N}$, then $\tilde{V}_1, \ldots$ is a sequence of i.i.d. uniform variables which is independent of $\tilde{B}$. Finally, introduce the random permutation $\tilde{\tau}$ of $\{1, \ldots, k\}$ such that for $i = 1, \ldots, k$, $\tilde{\tau}(i)$ is the index of the i-th interval amongst $\tilde{I}_1, \ldots, \tilde{I}_k$ visited by the sequence $\tilde{V}_1, \ldots$, and $\tilde{\pi}(\tilde{B})$ the partition of $\mathbb{N}$ with (non-empty)blocks

$$\tilde{A}_i = \left\{ n \in \mathbb{N} : \tilde{V}_n \in \tilde{I}_{\tilde{\tau}(i)} \right\} , \qquad i = 1, \ldots, k .$$

It is immediate to check that $\tau = \eta \circ \tilde{\tau}$ and $\tilde{\pi}(\tilde{B}) = \pi(B)$. Since $(\tau, \pi(B))$ and $(\tilde{\tau}, \tilde{\pi}(\tilde{B}))$ have the same law, this completes the proof of (6) and (7).

Now the statement (i) follows from (4), (5) and (6) and (ii) derives from (4), (5) and (7).

Let us turn to the continuous case. Then β is the limit in $\mathcal{S}$ of a sequence β_n with $\beta_n \in \mathcal{S}_f$ for every $n \in \mathbb{N}$. Denote by B_n a bridge whose ranked sequence of jumps is β_n. By Lemma 1, the sequence B_n converges in distribution to B, and by Skorohod's representation theorem we may even assume that B_n converges a.s. to B in the Skorohod topology. It easily follows that $\pi_n(B)$ converges a.s. to $\pi(B)$ (recall that we use the *same* sequence $V_1, V_2, \ldots$ to define $\pi_n(B)$ and $\pi(B)$) and B_n^{-1} converges to B^{-1} uniformly on $[0, 1]$, a.s. Finally the desired result follows by applying the discrete case to each bridge B_n and passing to the limit $n \to \infty$. $\qquad \square$

The following corollary of Lemma 2 is crucial for the applications that follow. Note that, if B and B' are two independent bridges, then the composition $B \circ B'$ is again a bridge, and simple arguments show that we have $(B \circ B')^{-1} = B'^{-1} \circ B^{-1}$ a.s.

Corollary 1 *Let $p \geq 2$ and let $B^1, \ldots, B^p$ be p independent bridges. For every $k \in \{1, \ldots, p\}$, let C^k be the bridge defined by $C^k = B^1 \circ \cdots \circ B^k$. Then conditionally on $(\pi(C^1), \ldots, \pi(C^{p-1}))$, $\pi(C^p)$ is distributed as the coagulation of $\pi(C^{p-1})$ by an independent exchangeable partition distributed as $\pi(B^p)$.*

Proof: We define a random partition $\tilde{\pi}(B^m)$ and a sequence $(V_i^{(m)}, i \in \mathbb{N})$ for every $m \in \{1, \ldots, p\}$ by induction on m. We first take $\tilde{\pi}(B^1) = \pi(B^1)$, and if $A_i^{(1)}, i \in \mathbb{N}$ denote the blocks of $\tilde{\pi}(B^1)$, we set

$$V_i^{(1)} = (B^1)^{-1}(V_j) , \; j \in A_i^{(1)},$$

and on the event where the number K_1 of blocks of $\pi(B^1)$ is finite, we define $V_{K_1+1}^{(1)}, V_{K_1+2}^{(1)}, \ldots$ as explained after (3). Suppose that for $m \in \{2, \ldots, p\}$ the random variables $V_j^{(m-1)}, j \in \mathbb{N}$ have been defined. We then let $\tilde{\pi}(B^m)$ be defined by

$$i \overset{\tilde{\pi}(B^m)}{\sim} j \iff (B^m)^{-1}(V_i^{(m-1)}) = (B^m)^{-1}(V_j^{(m-1)}) , \qquad i, j \in \mathbb{N}, \tag{8}$$

and, if $\widetilde{A}_i^{(m)}, i \in \mathbb{N}$ are the blocks of $\widetilde{\pi}(B^m)$, we also set

$$V_i^{(m)} = (B^m)^{-1}(V_j^{(m-1)}) \ , \ j \in \widetilde{A}_i^{(m)}, \tag{9}$$

with a similar convention when the number of blocks of $\widetilde{\pi}(B^m)$ is finite (each time we use different auxiliary independent uniform variables). Lemma 2 and an easy induction show that for every $m \in \{1, \ldots, p\}$, the variables $V_j^{(m)}, j \in \mathbb{N}$ are i.i.d. uniform over $[0,1]$. Furthermore, if $m \geq 2$, B^m is independent of $(V_j^{(m-1)}, j \in \mathbb{N})$, and it follows that $\widetilde{\pi}(B^m)$ has the same distribution as $\pi(B^m)$.

Let $m \in \{2, \ldots, p\}$. Using the fact that $(C^m)^{-1} = (B^m)^{-1} \circ \cdots \circ (B^1)^{-1}$, we observe that the distinct values in the sequence $((C^m)^{-1}(V_j), j \in \mathbb{N})$ appear, in the same order, as the first values of the sequence $(V_j^{(m)}, j \in \mathbb{N})$. Let $A_j^{(m)}, j \in \mathbb{N}$ be the blocks of the partition $\pi(C^m)$. It follows from the previous observation that, for every j such that $A_j^{(m)} \neq \emptyset$, and every $i \in \mathbb{N}$, $i \in A_j^{(m)}$ if and only if $(C^m)^{-1}(V_i) = V_j^{(m)}$.

Using this fact, the definition of $\pi(C^m)$ and the identity $(C^m)^{-1} = (B^m)^{-1} \circ (C^{m-1})^{-1}$, we obtain that i and i' are in the same block of $\pi(C^m)$ if and only if $(B^m)^{-1}(V_k^{(m-1)}) = (B^m)^{-1}(V_{k'}^{(m-1)})$, where k and k' are defined by $i \in A_k^{(m-1)}$, $i' \in A_{k'}^{(m-1)}$. Thus, $\pi(C^m)$ is the coagulation of $\pi(C^{m-1})$ by $\widetilde{\pi}(B^m)$.

Since $\widetilde{\pi}(B^m)$ has the same distribution as $\pi(B^m)$, the statement of the corollary will follow (taking $p = m$) if we can prove that $\widetilde{\pi}(B^m)$ is independent of $(\pi(C^1), \ldots, \pi(C^{m-1}))$. From the definition (8) of $\widetilde{\pi}(B^m)$, it is enough to verify that $(V_1^{(m-1)}, V_2^{(m-1)}, \ldots)$ is independent of $(\pi(C^1), \ldots, \pi(C^{m-1}))$. We prove by induction that for every $\ell \in \{1, \ldots, p\}$:

$$\text{The variables } V_j^{(\ell)}, \ j = 1, 2, \ldots \text{ are independent of } (\pi(C^1), \ldots, \pi(C^\ell)). \tag{10}$$

For $\ell = 1$, this follows from assertion (i) of Lemma 2. Let $\ell \in \{2, \ldots, p\}$ and assume that (10) holds at order $\ell - 1$, so that the variables $V_1^{(\ell-1)}, \ldots$ are independent of $(\pi(C^1), \ldots, \pi(C^{\ell-1}))$. It follows that $(B^\ell, (V_1^{(\ell-1)}, \ldots))$ is independent of $(\pi(C^1), \ldots, \pi(C^{\ell-1}))$. From (8), $\widetilde{\pi}(B^\ell)$ is a function of $(B^\ell, (V_1^{(\ell-1)}, \ldots))$, and by (9), $(V_1^{(\ell)}, \ldots)$ is a function of $(B^\ell, (V_1^{(\ell-1)}, \ldots))$ and, possibly, of auxiliary independent variables. Thus the pair $((V_1^{(\ell)}, \ldots), \widetilde{\pi}(B^\ell))$ is independent of $(\pi(C^1), \ldots, \pi(C^{\ell-1}))$. Finally, we also know from Lemma 2 (i) that $(V_1^{(\ell)}, \ldots)$ is independent of $\widetilde{\pi}(B^\ell)$. It follows that $(V_1^{(\ell)}, \ldots)$ is independent of $(\pi(C^1), \ldots, \pi(C^{\ell-1}), \widetilde{\pi}(B^\ell))$ and since we saw that $\pi(C^\ell)$ is the coagulation of $\pi(C^{\ell-1})$ by $\widetilde{\pi}(B^\ell)$, $(V_1^{(\ell)}, \ldots)$ is independent of $(\pi(C^1), \ldots, \pi(C^\ell))$. This completes the proof of (10) and of the corollary. $\square$

3 Exchangeable coalescents and flows of bridges

In this section, we present the basic correspondence between flows of bridges and exchangeable coalescents. This can be viewed as an infinite-dimensional extension of that between bridges and exchangeable partitions which we described at the beginning of Section 2.3.

3.1 Some definitions

We first define the notion of an exchangeable $\mathcal{P}$-coalescent.

Definition 1 *An exchangeable $\mathcal{P}$-coalescent is a Markov process $\Pi = (\Pi_t, t \geq 0)$ with values in $\mathcal{P}$, which is continuous in probability, and such that its semigroup $(P_t, t \geq 0)$ satisfies the following property: For every $t \geq 0$, there exists an exchangeable partition π_t such that for every $\eta \in \mathcal{P}$, $P_t(\eta, \cdot)$ is the law of the coagulation of η by π_t.*

We stress that the semigroup of an exchangeable $\mathcal{P}$-coalescent always enjoys the Feller property, thanks to the continuity of coagulation maps.

Let $\Pi = (\Pi_t, t \geq 0)$ be an exchangeable $\mathcal{P}$-coalescent. For every $t \geq 0$, let Π_t^n denote the restriction of the partition Π_t to $\{1, \ldots, n\}$. Then it is easy to verify that the process $(\Pi_t^n, t \geq 0)$ is also a Markov process, with values in the (finite) set $\mathcal{P}_n$ of all partitions of $\{1, \ldots, n\}$: Precisely, if P_t^n denotes the semigroup of Π^n and if $\eta \in \mathcal{P}_n$ has p blocks, $P_t^n(\eta, \cdot)$ is the distribution of the coagulation of η by the restriction of π_t to $\{1, \ldots, p\}$. The law of the Markov chain $(\Pi_t^n, t \geq 0)$ is then characterized by its transition rates. Thanks to exchangeability, we only have to consider, for every integer $p \geq 2$ and every $k_1, \ldots, k_m \geq 2$ with $k_1 + \cdots + k_m \leq p$, the rate $\beta_{p;k_1,\ldots,k_m}$ corresponding to the following transition. Starting from a partition π with k blocks, if $\mathcal{C}_1, \ldots, \mathcal{C}_m$ are disjoint subcollections of blocks of π containing respectively $k_1, \ldots, k_m$ blocks, a new partition π' is obtained by merging the blocks of $\mathcal{C}_i$, for every $i = 1, \ldots, m$, whereas the remaining $p - (k_1 + \cdots + k_m)$ blocks which are not in $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_m$ are left unchanged in π'. In this form, we see that the class of exchangeable $\mathcal{P}$-coalescents is exactly the class of coalescents recently studied by Schweinsberg [18]. See also the discussion in Section 3.3 of [15] or Section 2.6 of [16].

Coalescents with multiple collisions [15] correspond to the special case where $\beta_{p;k_1,\ldots,k_m} = 0$ whenever $m \geq 2$. Thus the only (possibly) nonzero rates are the $\beta_{p,k}$ for $2 \leq k \leq p$. According to [15], these rates must be of the form

$$\beta_{p,k} = \int \Lambda(dx)\, x^{k-2}\,(1-x)^{p-k}. \tag{11}$$

where Λ is a finite measure on $[0, 1]$. Conversely, given any finite measure Λ on $[0, 1]$, there is a coalescent with multiple collisions whose rates are given by (11), which is called the Λ-coalescent. We refer to the *standard* Λ-coalescent when the initial value is the partition of $\mathbb{N}$ in singletons.

We now introduce flows of bridges. We let Id denote the identity mapping $\mathrm{Id}(r) = r$ for every $r \in [0, 1]$.

Definition 2 *A flow of bridges is a collection $(B_{s,t}, -\infty < s \leq t < \infty)$ of bridges such that:*

(i) For every $s < t < u$, $B_{s,u} = B_{s,t} \circ B_{t,u}$ a.s.

(ii) The law of $B_{s,t}$ only depends on $t - s$. Furthermore, if $s_1 < s_2 < \cdots < s_n$, the bridges $B_{s_1,s_2}, B_{s_2,s_3}, \ldots, B_{s_{n-1},s_n}$ are independent.

(iii) $B_{0,0} = \text{Id}$ *and* $B_{0,t} \longrightarrow \text{Id}$ *in probability as* $t \downarrow 0$, *in the sense of Skorohod's topology.*

In (iii), the convergence $B_{0,t} \longrightarrow \text{Id}$ as $t \downarrow 0$ is equivalent to saying that $B_{0,t}(r)$ converges to r in probability for every $r \in [0,1]$.

If $(B_{s,t}, -\infty < s \leq t < \infty)$ is a flow of bridges, we can for each $t \geq 0$ construct the exchangeable partition $\pi(B_{0,t})$ as in subsection 2.3 for $B = B_{0,t}$. In particular, the ranked sequence of frequencies of $\pi(B_{0,t})$ coincides with that of the jump sizes of the bridge $B_{0,t}$. We stress that the same sequence of i.i.d. uniform variables $V_1, \ldots$ is used to define $\pi(B_{0,t})$ for all t's.

3.2 The basic correspondence

The next theorem shows that there is a one-to-one correspondence between distributions of flows of bridges and distributions of exchangeable $\mathcal{P}$-coalescents.

Theorem 1 (i) *Let* $(B_{s,t}, -\infty < s \leq t < \infty)$ *be a flow of bridges. Then the process* $(\pi(B_{0,t}), t \geq 0)$ *is an exchangeable* $\mathcal{P}$-*coalescent which starts from the partition of* $\mathbb{N}$ *into singletons.*

(ii) *Conversely, if* $(\Pi_t, t \geq 0)$ *is an exchangeable* $\mathcal{P}$-*coalescent started from the partition of* $\mathbb{N}$ *into singletons, then there is a flow of bridges* $(B_{s,t}, -\infty < s \leq t < \infty)$ *such that the processes* $(\pi(B_{0,t}), t \geq 0)$ *and* $(\Pi_t, t \geq 0)$ *have the same finite dimensional distributions.*

Proof: (i) Let $0 < t_1 < \cdots < t_n < t_{n+1}$. It follows from Corollary 1 that the conditional distribution of $\pi(B_{0,t_{n+1}})$ knowing $\pi(B_{0,t_1}), \pi(B_{0,t_2}), \ldots, \pi(B_{0,t_n})$ is the law of the coagulation of $\pi(B_{0,t_n})$ by an independent exchangeable partition distributed as $\pi(B_{t_n,t_{n+1}})$ or equivalently as $\pi(B_{0,t_{n+1}-t_n})$. This shows that the process $\pi(B_{0,t})$ is Markovian and that its semigroup satisfies the defining property of an exchangeable $\mathcal{P}$-coalescent. Finally, using the continuity lemma and the fact that $B_{0,t}$ converges in probability to the identity map Id as $t \to 0$, we get that $t \to \pi(B_{0,t})$ is continuous in probability.

(ii) Let $(\Pi_t, t \geq 0)$ be an exchangeable $\mathcal{P}$-coalescent started from the partition of $\mathbb{N}$ in singletons. With this special initial value, it is immediate that the exchangeable partition π_t of Definition 1 is given by $\pi_t = \Pi_t$. For every $t \geq 0$, let B_t be a bridge such that $\pi(B_t) \overset{d}{=} \Pi_t$ (note that B_t may be obtained from the explicit formula (2), letting β be the ranked sequence of frequencies of Π_t). Then let $s, t > 0$ and let B_s' be a copy of B_s independent of B_t. Corollary 1 implies that $\pi(B_t \circ B_s')$ has the same distribution as the coagulation of $\pi(B_t) \overset{d}{=} \Pi_t$ by an independent copy of $\pi(B_s') \overset{d}{=} \Pi_s$. It follows that $\pi(B_t \circ B_s') \overset{d}{=} \Pi_{s+t}$ and thus $B_t \circ B_s' \overset{d}{=} B_{s+t}$. We can now use the Kolmogorov extension theorem to construct a collection $(B_{s,t}, -\infty < s \leq t < \infty)$ of bridges such that, for every $s < t$, $B_{s,t} \overset{d}{=} B_{t-s}$ and $B_{s_1,s_2}, \ldots B_{s_{k-1},s_k}$ are independent whenever $s_1 < \cdots < s_k$. The continuity in probability of Π_t as $t \downarrow 0$ gives the analogous property for $B_{0,t}$ as $t \downarrow 0$. $\qquad \square$

Example 1: The Kingman coalescent. Consider the case when $(\Pi_t, t \geq 0)$ is the so-called Kingman coalescent. Recall from Theorem 4 in [12] that the process of the number $|\Pi_t|$ of blocks of Π_t is a pure death process with death rate $k(k-1)/2$ at level $k \in \mathbb{N}$. Moreover, Π_t is the trivial

partition of $\mathbb{N}$ into a single block if $|\Pi_t| = 1$, and conditionally on $|\Pi_t| = k \geq 2$, the sequence of the asymptotic frequencies of its blocks ranked in random order, is the uniform probability measure on the simplex $\{(x_1, \ldots, x_k) : x_i \geq 0 \text{ and } \sum_{i=1}^{k} x_i = 1\}$, i.e. the k-dimensional Dirichlet distribution with parameter $(1, \ldots, 1)$. It now follows from (2) that the law of the bridge $B_{0,t}$ which is associated to Kingman coalescent by Theorem 1, can be described as follows: The process $\#(t)$ of the number of jumps of $B_{0,t}$ is a pure death process with death rate $k(k-1)/2$ at level $k \in \mathbb{N}$, and started at $\#(0+) = \infty$; furthermore the conditional distribution of $B_{0,t}$ given $\#(t) = k$ is that of

$$\sum_{i=1}^{k} \beta^i \, 1_{\{U^i \leq r\}}, \qquad r \in [0,1],$$

where $U_1, \ldots, U_k$ is a sequence of k i.i.d. uniform $[0,1]$ variables, and $\beta = (\beta^1, \ldots, \beta^k)$ a Dirichlet variable with parameter $(1, \ldots, 1)$ which is independent of the U_i's (for $k = 1$, one has $\beta^1 = 1$ a.s.).

Example 2: The Bolthausen-Sznitman coalescent. For $\alpha \in]0,1[$, let $\sigma^{(\alpha)}$ denote a stable subordinator with index α. Then the process $(B^{(\alpha)}(r) = \sigma^{(\alpha)}(r)/\sigma^{(\alpha)}(1), r \in [0,1])$ is a bridge. Furthermore, if $\alpha, \beta \in]0,1[$ and $\widetilde{B}^{(\beta)}$ is a copy of $B^{(\beta)}$ independent of $B^{(\alpha)}$, the composition $B^{(\alpha)} \circ \widetilde{B}^{(\beta)}$ has the same distribution as $B^{(\alpha\beta)}$. It easily follows that there exists a flow of bridges $(B_{s,t}, -\infty < s \leq t < \infty)$ such that $B_{s,t}$ is distributed as $B^{(\alpha)}$ for $\alpha = e^{s-t}$. In this situation, $(\pi(B_{0,t}), t \geq 0)$ is the so-called Bolthausen-Sznitman coalescent which was introduced in [4]. See [1] and [3] for various results related to this example.

Remarks. (i) Many results about Λ-coalescents have a natural interpretation via Theorem 1, in the setting of flows of bridges. For instance, a bridge B has zero drift coefficient a.s. if and only if the exchangeable partition $\pi(B)$ has so-called proper frequencies, that is the total sum of the asymptotic frequencies of its blocks equals 1 a.s. (or equivalently, none of its blocks is a singleton a.s.). Theorem 8 of Pitman [15] provides an explicit characterization of this property for Λ-coalescents in terms of the measure Λ. Similarly, the discrete case holds for the bridge $B_{s,t}$ if and only if the associated coalescent comes down from infinity, which means that for every $t > 0$, Π_t has only finitely many non-empty blocks. We refer to Schweinsberg [19] for a characterization of this property for Λ-coalescents. More generally, these questions can also be addressed for bridges associated to arbitrary exchangeable $\mathcal{P}$-coalescents; see e.g. Proposition 30 in Schweinsberg [18].

(ii) For a general flow of bridges B, it is also natural to consider the family of inverse mappings $B_{s,t}^{-1}$ for $s \leq t$. It follows from the cocycle property

$$B_{s,u}^{-1} = B_{t,u}^{-1} \circ B_{s,t}^{-1}, \qquad \text{a.s. for every } s \leq t \leq u$$

that for every $x \in [0,1]$, the one-point motion $\left(B_{0,t}^{-1}(x)\right)_{t \geq 0}$ is Markovian; note from Lemma 2 that Lebesgue measure on $[0,1]$ is invariant for this Markov process. More generally, for every integer $k \geq 1$ and every $x_1, \ldots, x_k \in [0,1]$, the k-point motion $\left(B_{0,t}^{-1}(x_1), \ldots, B_{0,t}^{-1}(x_k)\right)_{t \geq 0}$ is also Markovian, and the Feller property is fulfilled whenever for each $t > 0$, the map $r \to B_{0,t}^{-1}(r)$ has no fixed discontinuity. Further, in the continuous case (i.e. when the map $r \to B_{0,t}^{-1}(r)$ is continuous a.s. for every $t \geq 0$), the Feller property still holds if we restrict the state space to the *open* interval $]0,1[$, and the family $\left(B_{s,t}^{-1}, -\infty < s \leq t < \infty\right)$ then induces a *coalescing*

stochastic flow on $]0, 1[$, in the terminology of Le Jan and Raimond [13] (the boundary points 0 and 1 have to be excluded as they are absorbing states).

3.3 Flow of bridges and mass coalescents

In this section, we briefly reformulate the preceding results in terms of the so-called mass coalescents (see e.g. Section 3.7 of [15]). Recall from section 2.1 the definition of the sets $\mathcal{S}$ and $\mathcal{S}_f$, and denote by 0 the sequence in $\mathcal{S}$ with all components equal to 0. If $x = (x_i)_{i\in\mathbb{N}} \in \mathcal{S}_f$ and π is a partition of $\mathbb{N}$, the coagulation of x by π is the element of $\mathcal{S}_f$ whose components are the numbers

$$\sum_{i\in A} x_i$$

when A varies over the blocks of π (these numbers must obviously be ranked in nonincreasing order, and if π has finitely many blocks, the other components are taken equal to 0).

Definition 3 *An exchangeable mass coalescent is a Markov process $X = (X_t, t \geq 0)$ with values in $\mathcal{S}$, with a Feller semigroup $(Q_t, t \geq 0)$ satisfying the following property: For every $t \geq 0$, there exists a random exchangeable partition π_t of $\mathbb{N}$, such that, for every $x \in \mathcal{S}_f$, $P_t(x, \cdot)$ is the law of the coagulation of x by π_t.*

Let X be an exchangeable mass coalescent with semigroup Q_t and let π_t be as in the definition above. Then for every $t > 0$ and every $x \in \mathcal{S}$, $Q_t(x, \cdot)$ is the distribution of the ranked sequence of the numbers

$$\sum_{i\in A} x_i + (1 - |x|)f_A,$$

when A varies over the blocks of π_t, and f_A is as previously the asymptotic frequency of A. This easily follows from the definition by approximating x by a sequence in $\mathcal{S}_f$ and using the Feller property of the semigroup.

There is a one-to-one correspondence between laws of exchangeable $\mathcal{P}$-coalescents and laws of exchangeable mass coalescents. On one hand, if $(\Pi_t, t \geq 0)$ is an exchangeable $\mathcal{P}$-coalescent and β_t is the ranked sequence of frequencies of Π_t, then the process $(\beta_t, t \geq 0)$ is an exchangeable mass coalescent, whose transition kernel $Q_t(x, \cdot)$ is the law of the coagulation of x by the exchangeable partition π_t appearing in Definition 1. Conversely, if X is an exchangeable mass coalescent and π_t is the (unique in law) exchangeable partition introduced in Definition 3, then for every $s, t > 0$, π_{t+s} has the same distribution as the coagulation of π_t by an independent copy of π_s. Hence there exists an exchangeable $\mathcal{P}$-coalescent $(\Pi_t, t \geq 0)$ such that $\Pi_t \overset{d}{=} \pi_t$ for every $t > 0$.

We can thus reformulate Theorem 1 as follows:

(i) If $B = (B_{s,t}, -\infty < s < t < \infty)$ is a flow of bridges and $\beta_t = (\beta_t^i, i \in \mathbb{N})$ denotes the ranked sequence of jump sizes of $B_{0,t}$, then the process $(\beta_t, t \geq 0)$ is an exchangeable mass coalescent with initial value $\beta_0 = 0$.

(ii) Conversely, if X is an exchangeable mass coalescent started at $X_0 = 0$, there exists a flow of bridges B such that the process $(\beta_t, t \geq 0)$ defined in (i) has the same distribution as X.

13

4   Poissonian construction of flows of bridges

It has been known from the pioneering works of Lévy and Itô that Poisson point processes provide a natural way for constructing a large class of stochastic flows. In this section, we shall first use this method to construct simple flows of bridges and specify the law of the associated $\mathcal{P}$-exchangeable coalescents. Then we shall show that flows of bridges corresponding to general Λ-coalescents arise as weak limits of such simple flows.

To start with, we introduce for every $x \in]0,1]$ and $u \in]0,1[$ the elementary function

$$b_{u,x}(r) = (1-x)r + x\mathbf{1}_{\{u \leq r\}}, \qquad r \in [0,1].\tag{12}$$

Note that if, as usual, U stands for a uniform $[0,1]$ variable, then the randomly mixed function $b_{U,x}$ is a prototype of a bridge. Next, consider a Poisson random measure M on $\mathbb{R} \times]0,1[\times]0,1]$ with intensity $dt \otimes du \otimes \nu(dx)$, where ν is some finite measure on $]0,1]$. With probability one, for every $s \leq t$ the number K of atoms of M on $]s,t] \times]0,1[\times]0,1]$ is finite; let $(t_1, u_1, x_1), \ldots, (t_K, u_K, x_K)$ denote these atoms with $s < t_1 < \cdots < t_K \leq t$. We then define

$$B_{s,t}^M = b_{u_1,x_1} \circ \cdots \circ b_{u_K,x_K},\tag{13}$$

where we agree that the right-hand side above is simply the identity map on $[0,1]$ when $K = 0$.

Lemma 3 $B^M = \left(B_{s,t}^M, -\infty < s \leq t < \infty\right)$ *is a flow of bridges.*

The flow B^M will be called a ν-*simple flow of bridges.*

Proof: All that we need to check is that $B_{s,t}^M$ is indeed a bridge, as the other required properties are obvious. To that end, let us first argue conditionally on the number K of atoms of M on $]s,t] \times]0,1[\times]0,1]$.

It is well-known that conditionally on $K = k \geq 1$, the variables $u_1, \ldots, u_k, x_1, \ldots, x_k$ are independent, and the u_i's are uniformly distributed on $]0,1[$. Hence $b_{u_1,x_1}, \ldots, b_{u_k,x_k}$ are independent bridges, and since the composition of two independent bridges is again a bridge, we see by induction that $B_{s,t}^M$ is a bridge. This is also the case conditionally on $K = 0$, since then $B_{s,t}^M \equiv \mathrm{Id}$. As mixing laws of bridges preserves the bridge property, the proof is complete. $\square$

Let us now describe the evolution of the exchangeable $\mathcal{P}$-coalescent $\Pi^M := (\pi(B_{0,t}^M), t \geq 0)$ which is constructed from the simple flow B^M as in Theorem 1. Plainly, Π^M is a step Markov process, and more precisely, $t > 0$ is a jump-time for Π^M only if the fiber $\{t\} \times]0,1[\times]0,1]$ contains an atom of the Poisson measure M.

Specifically, let (t,u,x) be, say, the k-th atom of M on $[0,\infty[\times]0,1[\times]0,1]$ and $B := B_{0,t-}^M$ the state of the flow immediately before t. Then, by the same argument as in the proof of Lemma 3, B is a bridge, and the partition $\pi = \pi(B)$ is the state of the coalescent Π^M immediately before the jump. The partition $\pi' = \Pi_t^M$ after the jump is obtained as follows: To each block A_i of π, we associate the variable V_i' defined by (3) and consider the set

$$I = \{i \in \mathbb{N} : (1-x)u \leq V_i' < (1-x)u + x\}.$$

Then π' is the partition whose blocks are the A_j's for $j \in \mathbb{N} \setminus I$ and $A' = \bigcup_{i \in I} A_i$, the block obtained by the coagulation of all the blocks with indices in I.

It is convenient to reformulate this description in terms of the variables

$$\xi_i \;=\; \mathbf{1}_{\{(1-x)u \leq V'_i < (1-x)u+x\}}\,, \qquad i \in \mathbb{N}\,.$$

Since by Lemma 2, $V'_1, \dots$ are i.i.d. uniform $[0,1]$ variables which are independent of π, u and x, we get that conditionally on x, the ξ_i's form an i.i.d. sequence of Bernoulli variables with $\mathbb{P}(\xi_i = 1) = x$, which is independent of π. We see that π' then results from the coagulation of all the blocks A_i with $\xi_i = 1$ into a single block. This last formulation enables us to identify Π^M as a standard Λ-coalescent, where Λ is the measure on $[0,1]$ given by $\Lambda(dx) = x^2\nu(dx)$: See Corollary 3 in Pitman [15], and also Sagitov [17].

We record this analysis in the following statement.

Lemma 4 *Let ν be a finite measure on $]0,1]$. Let B be a ν-simple flow of bridges and let $\Pi = (\pi(B_{0,t}), t \geq 0)$ be the exchangeable $\mathcal{P}$-coalescent associated with B by Theorem 1. Then Π is a standard Λ-coalescent for $\Lambda(dx) = x^2\nu(dx)$.*

We now state a result of convergence in distribution of a sequence of simple flows of bridges, which yields a Poissonian construction of flows of bridges corresponding to the Pitman-Sagitov Λ-coalescents.

Theorem 2 *Consider a sequence $(\nu_n, n \in \mathbb{N})$ of finite measures on $]0,1]$, and for every n, let $(B^n_{s,t}, -\infty < s \leq t < \infty)$ be a ν_n-simple flow of bridges. Suppose that $\Lambda_n(dx) := x^2\nu_n(dx)$ converges weakly to some finite measure $\Lambda(dx)$ on $[0,1]$. Then as $n \to \infty$, $(B^n_{s,t}, -\infty < s \leq t < \infty)$ converges in the sense of weak convergence of finite dimensional distributions to the flow of bridges $(B_{s,t}, -\infty < s \leq t < \infty)$ such that the associated exchangeable $\mathcal{P}$-coalescent $(\pi(B_{0,t}), t \geq 0)$ is a standard Λ-coalescent.*

Proof: We know from Theorem 1 of Pitman [15] that the standard Λ_n-coalescent converges in distribution to the standard Λ-coalescent. By Lemma 4 and the continuity lemma, this readily entails that for every $s < t$, $B^n_{s,t}$ converges in distribution to $B_{s,t}$, where B is the flow of bridges associated with the Λ-coalescent by Theorem 1. The convergence of finite-dimensional marginals then follows using the independence properties of flows of bridges. $\qquad\square$

If we start from a given finite measure Λ on $[0,1]$, we can find a sequence ν_n of finite measures on $]0,1]$ such that the measures $x^2\nu_n(dx)$ converge weakly to Λ. Lemma 4 and Theorem 2 then provide a (weak) Poissonian construction of the flow of bridges corresponding to the Λ-coalescent. By analogy with the case of Lévy processes, it is tempting to reinforce this construction to a strong one, that is to obtain an almost sure version of the convergence of Theorem 2.

For this purpose, let us consider a finite measure Λ on $[0,1]$ with $\Lambda(\{0\}) = 0$, and let M be a Poisson random measure on $\mathbb{R} \times]0,1[\times]0,1]$ with intensity $dt \otimes du \otimes x^{-2}\Lambda(dx)$. For every $n \in \mathbb{N}$, the restriction M^n of M to $\mathbb{R} \times]0,1[\times]1/n,1]$ is a Poisson measure with intensity $dt \otimes du \otimes \nu_n(dx)$, where ν_n is the finite measure given by $\nu_n(dx) = \mathbf{1}_{\{x>1/n\}}x^{-2}\Lambda(dx)$. We can thus use (13) to construct a ν_n-simple flow of bridges B^n from the Poisson measure M^n, and since $\Lambda_n(dx) := x^2\nu_n(dx)$ converges weakly to $\Lambda(dx)$, Theorem 2 shows that B^n converges

weakly as $n \to \infty$ towards the flow of bridges corresponding to the Λ-coalescent. Under the additional condition

$$\int_{[0,1]} x^{-1} \Lambda(dx) < \infty \,. \tag{14}$$

we can in fact get an almost sure convergence. Assume further for simplicity that $\int x^{-2} \Lambda(dx) = \infty$ (as otherwise there is no difficulty) and $\Lambda(\{1\}) = 0$. For fixed $s < t$, denote by $\{(t_i, u_i, x_i), i \in \mathbb{N}\}$ the sequence of the atoms of the Poisson measure M on $]s, t] \times]0, 1[\times]0, 1]$, ranked in the decreasing order of the third coordinate. On the one hand, for every $u, x \in]0, 1[$, the inverse $b_{u,x}^{-1}$ of the elementary function $b_{u,x}$, is Hölder continuous with parameter $1/(1 - x)$. On the other hand, assumption (14) ensures that with probability one,

$$\sum_{i \in \mathbb{N}} x_i < \infty \,,$$

It is then a simple exercise to verify that $(B_{s,t}^n)^{-1}$ converges in the sense of the uniform norm as $n \to \infty$, a.s. Therefore, in this special case, the limiting flow of Theorem 2 can be constructed in such a way that $(B_{s,t}^n)^{-1}$ converges uniformly to $(B_{s,t})^{-1}$, a.s., for every $s < t$. It is an interesting question whether such an almost sure convergence still holds when (14) fails.

We mention that the Poissonian approach developed in this section, can be extended to the construction of the most general flow of bridges (corresponding to the general exchangeable $\mathcal{P}$-coalescent studied by Möhle and Sagitov [14] and Schweinsberg [18]). Roughly, one needs to replace the elementary functions $b_{u,x}$ by more general functions the type

$$r \to \left(1 - \sum_{i=1}^{\infty} x_i\right) r + \sum_{i=1}^{\infty} x_i \mathbf{1}_{\{r \geq u_i\}} \,,$$

where $x_1 \geq x_2 \geq \cdots \geq 0$ is a sequence in $\mathcal{S}$ and $(u_i, i \in \mathbb{N})$ a sequence of distinct points in $]0, 1[$. Note that such functions correspond to general bridges conditioned by the sequences of their jump sizes and jump times. Details are left to the interested reader.

5 The dual flow

5.1 Dual flows and generalized Fleming-Viot processes

Here, we turn our attention to the *dual flow* $(\hat{B}_{s,t}, -\infty < s \leq t < \infty)$ defined by

$$\hat{B}_{s,t} = B_{-t,-s} \,,$$

where $(B_{s,t}, -\infty < s \leq t < \infty)$ is as in subsection 2.1. For the sake of simplicity, we shall assume that the flow is not trivial, in the sense that the case when $B_{s,t} \equiv \mathrm{Id}$ will be implicitly excluded from now on.

Plainly, $\hat{B}_{s,t}$ is a bridge whose law only depends on $t - s$, and which converges in probability to Id when $t - s \to 0$. Furthermore the bridges $\hat{B}_{s_1,s_2}, \ldots, \hat{B}_{s_{n-1},s_n}$ are independent for every $s_1 < \cdots < s_n$, and the following cocycle property holds for every $s < t < u$:

$$\hat{B}_{t,u} \circ \hat{B}_{s,t} = \hat{B}_{s,u} \,.$$

Recall that $\mathcal{M}_1$ denotes the space of all probability measures on $[0,1]$, which is equipped with the weak topology. We are especially interested in the process $(\rho_t)_{t \geq 0}$ taking values in $\mathcal{M}_1$, which is defined by the distribution function

$$\rho_t([0,y]) = \hat{B}_{0,t}(y) \quad \text{for every } y \in [0,1].$$

From the cocycle property for $\hat{B}$, we immediately get that this process is Markovian, with a transition semigroup R_t which can be described as follows. If $\mu \in \mathcal{M}_1$, and $F_{(\mu)}$ denotes the distribution function of μ, $R_t(\mu, \cdot)$ is the law of the random element of $\mathcal{M}_1$ whose distribution function is $B_{0,t} \circ F_{(\mu)}$. From this description and the fact that $B_{0,t}$ has no fixed discontinuities, we also see that $(\rho_t)_{t \geq 0}$ is a Feller process and in particular it has a càdlàg modification. From now on, we *always deal* with this càdlàg modification. Note that $\rho_0 = \lambda$ is Lebesgue measure on $[0,1]$. In the genealogical interpretation of Section 1, $\rho_t(dr)$ represents the size of the progeny at time t of the fraction dr of the initial population.

We may also consider, for a fixed $y \in [0,1]$, the process $F_t(y)$ defined by

$$F_t(y) = \rho_t([0,y]).$$

Again from the cocycle property, we see that this process is a Markov process, and that its semigroup Q_t does not depend on y: For every $x \in [0,1]$, $Q_t(x, \cdot)$ is simply the law of $B_{0,t}(x)$. This semigroup is again Feller thanks to the absence of fixed discontinuities for $B_{0,t}$. It is easy to verify that $\rho_t(\{y\}) = 0$ for every $t \geq 0$, a.s. (if $\varepsilon > 0$ and $T_\varepsilon = \inf\{t \geq 0 : \rho_t(\{y\}) \geq \varepsilon\}$, apply the strong Markov property of ρ at time T_ε to see that on the event $\{T_\varepsilon < \infty\}$ there will exist rational values of t for which $\rho_t(\{y\}) > 0$, which does not occur with probability one). From the right-continuity of paths of ρ_t, we now deduce that the paths of $F_t(y)$ are also right-continuous a.s. (they will indeed be càdlàg by the Feller property). The process $F_t(y)$ will be studied in subsection 5.3.

We will now argue that the process ρ_t can be viewed as a generalized Fleming-Viot process. Consider first the case of simple flows. Let $B_{s,t}^M$ be defined by (13), where M is a Poisson random measure on $\mathbb{R} \times]0,1[\times]0,1]$ with intensity $dt \otimes du \otimes \nu(dx)$ and ν is some finite measure on $]0,1]$. As above, we consider the process ρ^M with values in $\mathcal{M}_1$ defined for $t \geq 0$ by

$$\rho^M([0,y]) = \hat{B}_{0,t}^M(y), \qquad y \in [0,1],$$

and recall that

$$\hat{B}_{0,t}^M = b_{u_k,x_k} \circ \cdots \circ b_{u_1,x_1} \tag{15}$$

where $(-t_1, u_1, x_1), \ldots, (-t_k, u_k, x_k)$ are the atoms of M on $[-t, 0[\times]0,1[\times]0,1]$ with $0 < t_1 < \cdots < t_k \leq t$.

The process $(\rho_t^M, t \geq 0)$ is a continuous-time Markov chain, and we see from (15) that its jumps occur with rate $\nu(]0,1])$, independently of the state. More precisely, as the elementary function $b_{u,x}$ corresponds to the distribution of the probability measure $(1-x)dy + x\delta_u(dy)$ on $[0,1]$ (where δ_u stands for the Dirac point mass at u), if t is a jump time for ρ^M, then the conditional law of ρ_t^M given ρ_{t-}^M is that of

$$(1-X)\rho_{t-}^M + X\delta_Y,$$

where X and Y are independent random variables, X is distributed according to $\nu(\cdot)/\nu(]0,1])$ and Y according to ρ_{t-}^M.

This description bears obvious similarities with that for the evolution of the Fleming-Viot process; see e.g. Chapter 1 of Etheridge [7]. For this reason, we call ρ^M the ν-generalized Fleming-Viot process (ν-GFV process in short). A limit theorem for sequences of these simple generalized Fleming-Viot processes is easily deduced from Theorem 2. We let (ν_n) be a sequence of finite measures on $]0,1]$, and for every n we denote by $\rho^n = (\rho_t^n, t \geq 0)$ the ν_n-GFV process started at λ.

Corollary 2 *Let Λ be a finite measure on $[0,1]$. Assume that the finite measures $x^2\nu_n(dx)$ converge weakly to Λ. Then the sequence of processes (ρ^n) converges in distribution, in the sense of weak convergence of finite-dimensional marginals, to the process ρ corresponding to the flow of bridges B associated with the Λ-coalescent.*

This immediately follows from Theorem 2: If B^n denotes a ν_n-simple flow of bridges, and if $0 < t_1 < \cdots < t_k$, Theorem 2 yields the convergence in distribution of the k-tuple $(B_{-t_k,0}^n, \ldots, B_{-t_1,0}^n)$ towards $(B_{-t_k,0}, \ldots, B_{-t_1,0})$, which gives (and is in fact stronger than) the convergence of $(\rho_{t_1}^n, \ldots, \rho_{t_k}^n)$ towards $(\rho_{t_1}, \ldots, \rho_{t_k})$.

In subsection 5.2 below, we will see that the law of the process (ρ_t) is characterized by a martingale problem for which uniqueness is proved via a duality argument. (This approach also shows that the convergence in distribution in Corollary 2 holds in the sense of weak convergence in the Skorohod space $\mathbb{D}([0,\infty[,\mathcal{M}_1)$.) The process (ρ_t) appears as a measure-valued dual of the Pitman-Sagitov coalescents with multiple collisions. In this direction, it is interesting to recall the known connection between the standard Fleming-Viot process and Kingman's coalescent, see e.g. Section 5.2 of [7]. As mentioned above, these generalized Fleming-Viot processes can be viewed as continuous versions of Cannings' population model (see Section 1). Various properties motivated by this genealogical interpretation are discussed in subsection 5.3.

5.2 A martingale problem and a duality relation

In this section, we will provide a different approach to the Poisson construction of Theorem 2, based on the study of the martingale problem for the process ρ introduced in subsection 5.1. In the case of a simple flow of bridges, the description of the continuous-time Markov chain ρ^M given in subsection 5.1 shows that its law is characterized by the following martingale problem: For any bounded measurable function G on $\mathcal{M}_1$,

$$G(\rho_t^M) - \int_0^t ds \int \nu(dy) \int \rho_s^M(da) \Big(G\left((1-y)\rho_s^M + y\delta_a\right) - G\left(\rho_s^M\right) \Big) \tag{16}$$

is a martingale.

We will consider functions G of the following special type. Let $p \geq 1$ be an integer and let f be a continuous function on $[0,1]^p$. For every $\mu \in \mathcal{M}_1$, we set

$$G_f(\mu) = \int_{[0,1]^p} \mu(dx_1) \ldots \mu(dx_p) \, f(x_1, \ldots, x_p).$$

Let Λ be a finite measure on $[0,1]$, and let $\beta_{p,k}$ be the transition rates of the Λ-coalescent, as defined by (11). We introduce an operator L acting on functions of the type G_f:

$$LG_f(\mu) = \sum_{I \subset \{1,\dots,p\}, |I| \geq 2} \beta_{p,|I|} \int \mu(dx_1)\dots\mu(dx_p)\Big(f(R_I(x_1,\dots,x_{p-|I|+1})) - f(x_1,\dots,x_p)\Big). \quad (17)$$

Here, for every subset I of $\{1,\dots,p\}$ with $|I| \geq 2$, $R_I(x_1,\dots,x_{p-|I|+1})$ is defined as follows. If $\inf(I) = \ell$,

$$R_I(x_1,\dots,x_{p-|I|+1}) = (y_1,\dots,y_p)$$

where $y_i = x_\ell$ for $i \in I$ and the values $y_i, i \notin I$ listed in the order of $\{1,\dots,p\}\backslash I$ are the numbers $x_1,\dots,x_{\ell-1},x_{\ell+1},\dots,x_{p-|I|+1}$.

Let B be the flow of bridges associated with the Λ-coalescent and let ρ_t be defined from B as in subsection 5.1.

Theorem 3 (i) *The law of the process $(\rho_t, t \geq 0)$ is characterized by the following martingale problem. For every integer $p \geq 1$ and every continuous function f on $[0,1]^p$,*

$$G_f(\rho_t) - \int_0^t ds\, LG_f(\rho_s)$$

is a martingale.

(ii) *Let ν_n be a sequence of finite measures on $]0,1]$, and for every n let ρ^n be the Markov chain in $\mathcal{M}_1$ associated with ν_n, started at $\theta_n \in \mathcal{M}_1$. Assume that the finite measures $x^2\nu_n(dx)$ converge weakly to Λ and that the sequence θ_n converges weakly to θ_∞. Then the sequence of processes (ρ^n) converges in distribution in the sense of weak convergence for the Skorohod topology on $\mathbb{D}([0,\infty[, \mathcal{M}_1)$, to the process ρ started at θ_∞.*

The proof will show that the process ρ is a measure-valued dual to the Λ-coalescent, in the sense of Ethier and Kurtz [8] (Section 4.4).

Proof: We prove (i) and (ii) simultaneously. So let (ν_n) and (ρ^n) be as in (ii) and let φ be a bounded measurable function on $[0,1]$. Taking $G(\mu) = \langle \mu, \varphi \rangle$ and then $G(\mu) = \langle \mu, \varphi \rangle^2$ in (16), we get that for every n, both $\langle \rho_t^n, \varphi \rangle$ and

$$\langle \rho_t^n, \varphi \rangle^2 - \Big(\int \nu_n(dy)y^2\Big) \int_0^t ds\Big(\langle \rho_t^n, \varphi^2 \rangle - \langle \rho_t^n, \varphi \rangle^2\Big)$$

are martingales. Since the sequence $\int \nu_n(dy)y^2$ is bounded, a classical criterion (see e.g. Theorem VI.4.13 in [9]) ensures that the sequence of the laws of the processes $(\langle \rho_t^n, \varphi \rangle)_{t \geq 0}$ in $\mathbb{D}([0,\infty[, \mathbb{R})$ is tight. A standard argument then yields the tightness of the laws of the processes ρ^n in $\mathbb{D}([0,\infty[, \mathcal{M}_1)$.

At least along a subsequence, we can thus assume that the processes ρ^n converge weakly to some process ρ with càdlàg paths and values in $\mathcal{M}_1$. (Of course, in the special case where $\theta_n = \lambda$ for every n, we know from Corollary 2 that this process is the same as the one of subsection 5.1.) We again apply (16), now with

$$G(\mu) = \prod_{i=1}^p \langle \mu, \varphi_i \rangle$$

19

where φ_i is continuous on $[0,1]$, for every $i = 1,\ldots,p$. It follows that, for every n, $G(\rho_t^n) - \int_0^t ds\, L^n G(\rho_s^n)$ is a martingale, if

$$
\begin{aligned}
L^n G(\mu) &= \int \nu_n(dy) \int \mu(da) \Big(G((1-y)\mu + y\delta_a) - G(\mu) \Big) \\
&= \int \nu_n(dy) \int \mu(da) \Big(\prod_{i=1}^p \big((1-y)\langle \mu, \varphi_i \rangle + y\varphi(a) \big) - \prod_{i=1}^p \langle \mu, \varphi_i \rangle \Big) \\
&= \int \nu_n(dy) \Bigg(\sum_{I \subset \{1,\ldots,p\}} (1-y)^{p-|I|} y^{|I|} \Big(\prod_{i\notin I} \langle \mu, \varphi_i \rangle \Big) \langle \mu, \prod_{i\in I} \varphi_i \rangle - \prod_{i=1}^p \langle \mu, \varphi_i \rangle \Bigg) \\
&= \sum_{I \subset \{1,\ldots,p\},|I|\geq 2} \beta_{p,|I|}^n \Bigg(\Big(\prod_{i\notin I} \langle \mu, \varphi_i \rangle \Big) \langle \mu, \prod_{i\in I} \varphi_i \rangle - \prod_{i=1}^p \langle \mu, \varphi_i \rangle \Bigg),
\end{aligned}
$$

where $\beta_{p,k}^n = \int \nu_n(dy) y^k (1-y)^{p-k}$ for $2 \leq k \leq p$. The assumption of the theorem ensures that $\beta_{p,k}^n$ converges as $n \to \infty$ towards $\beta_{p,k}$. By passing to the limit $n \to \infty$, it follows that, for $f(x_1,\ldots,x_p) = \varphi_1(x_1)\cdots\varphi_p(x_p)$,

$$
G_f(\rho_t) - \int_0^t ds\, LG_f(\rho_s)
$$

is a martingale, where LG_f is as in the statement of the theorem. Since any continuous function on $[0,1]^p$ is the uniform limit of linear combinations of functions of the type $\varphi_1(x_1)\cdots\varphi_p(x_p)$, we easily conclude that the process ρ satisfies the martingale problem of the statement of the theorem.

We then verify that the law of ρ is characterized by this martingale problem. To this end we use a duality argument. Denote by $(\Pi_t^p)_{t\geq 0}$ the Λ-coalescent in $\mathcal{P}_p$. Following the description given in Section 3, its generator L^* can be written as follows: If $\pi \in \mathcal{P}_p$ is a partition with say q blocks $A_1,\ldots,A_q$, and if F is any function defined on $\mathcal{P}_p$,

$$
L^* F(\pi) = \sum_{J \subset \{1,\ldots,q\},|J|\geq 2} \beta_{q,|J|} \Big(F(c_J \pi) - F(\pi) \Big)
$$

where $c_J \pi$ denotes the new partition obtained from π by merging the blocks $A_i, i \in J$.

If f is a continuous function on $[0,1]^p$, we denote by Φ_f the function defined on $\mathcal{M}_1 \times \mathcal{P}_p$ by

$$
\Phi_f(\mu, \pi) = \int \mu(dx_1)\ldots\mu(dx_{|\pi|}) f(Y(\pi; x_1,\ldots,x_{|\pi|})),
$$

where, if π has $q = |\pi|$ blocks $A_1,\ldots,A_q$ listed in the order of their least elements, we put $Y(\pi; x_1,\ldots,x_q) = (y_1,\ldots,y_p)$, with $y_j = x_i$ if and only if $j \in A_i$.

If π is fixed, the function $\mu \longrightarrow \Phi_f(\mu, \pi)$ can be written in the form $G_g(\mu)$ for a certain continuous function g on $[0,1]^{|\pi|}$ and we can thus define $L\Phi_f(\mu, \pi) = LG_g(\mu)$. Similarly, if μ is fixed, we can view $\Phi_f(\mu, \cdot)$ as a function on $\mathcal{P}_p$, and define $L^*\Phi_f(\mu, \pi)$. The following lemma is easily deduced from the explicit form of L and L^*.

Lemma 5 *For every $\mu \in \mathcal{M}_1$ and $\pi \in \mathcal{P}_p$,*

$$
L\Phi_f(\mu, \pi) = L^*\Phi_f(\mu, \pi).
$$

This lemma establishes the duality between the limiting process ρ and the Λ-coalescent. Let π_0 denote the partition of $\{1, \ldots, p\}$ in singletons. Arguing as in Section 4.4 of [8], we infer from Lemma 5 that, for any solution ρ of the martingale problem of the theorem,

$$\mathbb{E}\left(G_f(\rho_t)\right) = \mathbb{E}\left(\Phi_f(\rho_t, \pi_0)\right) = \mathbb{E}\left(\Phi_f(\rho_0, \Pi_t^p)\right), \tag{18}$$

where Π_t^p starts from π_0 and it is implicit that ρ_0 and $(\Pi_s^p)_{s \geq 0}$ are independent. Hence the law of ρ_t is uniquely determined by that of ρ_0. From Theorem 4.4.2 in [8], this implies uniqueness for the martingale problem, as well as the strong Markov property for the solution. Parts (i) and (ii) of the theorem now follow. $\qquad\square$

Remarks. (1) The duality relation of Lemma 5 is of course reminiscent of that for the Fleming-Viot process. In the case when $\Lambda = c\,\delta_0$ (so that Π_t is the Kingman coalescent), $(\rho_t)_{t \geq 0}$ is a Fleming-Viot process without mutation, and we recover the duality presented e.g. in Section 1.12 of [7]. Note that duality for the Fleming-Viot process is usually presented in terms of a function-valued process rather than a coalescent.

(2) In the special case where $\rho_0 = \lambda$ is Lebesgue measure on $[0, 1]$, we know that ρ_t is distributed as

$$\left(1 - \sum_{i \in \mathbb{N}} \beta_t^i\right)\lambda + \sum_{i \in \mathbb{N}} \beta_t^i\,\delta_{U_i}$$

where $\beta_t = (\beta_t^i)_{i \in \mathbb{N}}$ is the ranked sequence of frequencies of the standard Λ-coalescent at time t, and $(U_i)_{i \in \mathbb{N}}$ are i.i.d. uniform on $[0, 1]$ and independent of β_t. It is a simple matter to recover this identity in distribution from the duality relation (18).

(3) Let ρ^μ denote the process ρ started at $\mu \in \mathcal{M}_1$. Then ρ^μ can be constructed from the special case $\mu = \lambda$ by the explicit formula

$$\rho_t^\mu([0, x]) = \rho_t^\lambda([0, \mu([0, x])]), \quad x \in [0, 1], \ t \geq 0.$$

This formula obviously corresponds to the description of the semigroup that was given in subsection 5.1.

(4) We could have proved directly the duality relation (18), without using the above calculations of generators. Consider first the case when $\rho_0 = \lambda$. Then $\Phi_f(\rho_t, \pi_0)$ has the same law as

$$\int_{[0,1]^p} dB_{0,t}(x_1) \cdots dB_{0,t}(x_p) f(x_1, \ldots, x_p) = \int_{[0,1]^p} dx_1 \cdots dx_p f(B_{0,t}^{-1}(x_1), \ldots, B_{0,t}^{-1}(x_p)).$$

In particular, taking expectations, we get

$$\mathbb{E}\left(\Phi_f(\rho_t, \pi_0)\right) = \mathbb{E}\left(f(B_{0,t}^{-1}(V_1), \ldots, B_{0,t}^{-1}(V_p))\right),$$

where $V_1, \ldots, V_p$ are i.i.d. uniform $[0, 1]$ variables which are independent of the bridge $B_{0,t}$. In the notation of Section 2.3 with $B = B_{0,t}$ and $\Pi_t = \pi(B_{0,t})$, we thus have

$$\mathbb{E}\left(\Phi_f(\rho_t, \pi_0)\right) = \mathbb{E}\left(f\left(Y(\Pi_t^p; V_1', \ldots, V_{|\Pi_t^p|}')\right)\right), \tag{19}$$

where Π_t^p stands for the restriction of the partition Π_t to $\{1, \ldots, p\}$. Recall from Lemma 3 that Π_t and $(V_1', \ldots)$ are independent and that the latter is a sequence of i.i.d. uniform $[0,1]$ variables. We now see that the right-hand side in (19) coincides with

$$\mathbb{E}\left(\Phi_f(\lambda, \Pi_t^p)\right).$$

This completes the proof of (18) in the case $\rho_0 = \lambda$, since we already know from Theorem 2 that $(\Pi_t, t \geq 0)$ is the standard Λ-coalescent. In the general case, simply use remark (3) above.

5.3   One point motion and the primitive Eve

We consider again the dual flow $(\hat{B}_{s,t}, -\infty < s \leq t < \infty)$ of a general flow of bridges and the process $(\rho_t, t \geq 0)$ with $\rho_0 = \lambda$, defined in subsection 5.1. In this section, we shall be interested in the one-point motions $\hat{B}_{s,t}(y)$, where $y \in [0,1]$ and $s \in \mathbb{R}$ are fixed, and $t \geq s$. By stationarity of the flow, there is no loss of generality in assuming $s = 0$, and we recall the notation
$$F_t(y) = \rho_t([0,y]) = \hat{B}_{0,t}(y).$$

We saw in subsection 5.1 that $F_t(y)$ is a Markov process with càdlàg paths and a Feller semi-group Q_t.

Proposition 1 *The process $(F_t(y), t \geq 0)$ is a martingale which converges a.s. to 0 or 1. More precisely,*
$$\mathbb{P}\left(\lim_{t \to \infty} F_t(y) = 1\right) = y.$$

Proof: The martingale property stems from the fact that for any bridge B, one has $\mathbb{E}(B(y)) = y$. Now the martingale $F_t(y)$ is bounded and thus converges a.s., and we have to check that the only possible limit points are 0 and 1.

The dynamics of coalescents implies that the probability that 1 and 2 belong to distinct blocks of the partition $\pi(B_{0,t})$ is e^{-ct}, where c is some positive real number (c has to be strictly positive as we have excluded the case when $B_{s,t} \equiv \mathrm{Id}$). Expressing this probability in terms of jump-sizes $\beta^i(t)$ of $B_{0,t}$, we get

$$\mathbb{E}\left(\sum_{i=1}^{\infty}(\beta^i(t))^2\right) = 1 - e^{-ct},$$

a quantity which converges to 1 when $t \to \infty$. Hence the ranked sequence of jump-sizes of $B_{0,t}$ converge in probability to $(1, 0, \ldots)$, and thus, by the continuity lemma, $B_{0,t}$ (or, equivalently, $B_{-t,0}$) converges in distribution to the bridge that has a unique jump with size one. The rest of the proof is now straightforward. $\qquad\square$

Proposition 1 suggests to investigate the set of initial values $y \in [0,1]$ for which $\lim_{t\to\infty} F_t(y) = 1$. By the monotonicity of bridges, this set is necessarily an interval. This yields the following definition:

Definition 4 *The random point*

$$\mathbf{e} := \inf\left\{y \in [0,1] : \lim_{t \to \infty} F_t(y) = 1\right\} = \sup\left\{y \in [0,1] : \lim_{t \to \infty} F_t(y) = 0\right\}$$

is called the primitive Eve.

Note that Proposition 1 entails that the primitive Eve is always uniformly distributed on $[0,1]$.

The terminology is motivated by the genealogical interpretation of the dual flow recalled in subsection 5.1. It will follow from Theorem 4 below that $\rho_t(\{\mathbf{e}\}) = F_t(\mathbf{e}) - F_t(\mathbf{e}-)$ converges to 1 as $t \to \infty$. However, the jump $F_t(y) - F_t(y-) = \rho_t(\{y\})$ corresponds to the size of the progeny at time t of the individual y of the initial generation. Thus, informally, the primitive Eve is the common ancestor at the initial generation of most of the individuals at time t when t is sufficiently large.

It is now natural to consider the process of the size of the progeny of the primitive Eve in the population at time t. To describe its distribution, we shall distinguish two cases. Write d_t for the drift coefficient of F_t. Then the cocycle property implies that $d_{t+s} = d_t \widetilde{d}_s$, where $\widetilde{d}_s$ is a copy of d_s independent of $(d_r)_{r \leq t}$. Using also the Feller property of the semigroup of (ρ_t), it follows that:

(i) Either $d_t = 0$ a.s., for every $t > 0$.

(ii) Either $\mathbb{P}(d_t > 0) > 0$ for every $t \geq 0$. Then $d_t = e^{-A_t}$, where $(A_t)_{t \geq 0}$ is a subordinator with values in $[0, \infty]$.

We refer to (i) as the *proper* case (since then the partitions $\pi(B_{0,t})$ have proper frequencies a.s. for every $t \geq 0$) and to (ii) as the *improper* case. Necessary and sufficient conditions for the proper case to hold can be found in Schweinsberg [18]; see also Theorem 8 in Pitman [15] for the case of Λ-coalescents.

Theorem 4 (i) *The process*

$$\rho_t(\{\mathbf{e}\}) = F_t(\mathbf{e}) - F_t(\mathbf{e}-), \qquad t \geq 0$$

is a Markov process in $[0,1]$ with a Feller semigroup Q_t^* given as follows: If $x > 0$,

$$\int Q_t^*(x, dy)\,\varphi(y) = \frac{1}{x}\,\mathbb{E}\Big(F_t(x)\,\varphi(F_t(x))\Big), \tag{20}$$

and

$$\int Q_t^*(0, dy)\,\varphi(y) = \mathbb{E}\Big(\int \rho_t(dy)\,\varphi(\rho_t(\{y\}))\Big). \tag{21}$$

In particular, the restriction to $]0,1]$ of the semigroup Q_t^* is the h-transform of Q_t for $h(x) = x$.

(ii) *In the proper case, 0 is an instantaneous point for the process $(\rho_t(\{\mathbf{e}\}), t \geq 0)$. In the improper case, 0 is a holding point and more precisely $\mathbb{P}(\rho_t(\{\mathbf{e}\}) = 0 \mid \rho_t) = d_t$ for every $t \geq 0$.*

Remark. In the proper case, formula (21) shows that the law of $\rho_t(\{\mathbf{e}\})$ is that of a size-biased sample of the atoms of ρ_t.

Proof: We first compute the distribution of $\rho_t(\{\mathbf{e}\})$ for a fixed $t > 0$. For every $s \geq 0$, set $\tilde{F}_s = B_{-t-s,-t}$. Then $(\tilde{F}_s, s \geq 0)$ is independent of F_t and has the same law as $(F_s, s \geq 0)$. Furthermore by the cocycle relation we have $\tilde{F}_s \circ F_t = F_{t+s}$ a.s., for every $s \geq 0$. Denote by $\tilde{\mathbf{e}}$ the primitive Eve for $\tilde{F}$. Let y be any rational in $[0, 1]$. On the event $\{y > \mathbf{e}\}$, we know that $\tilde{F}_s \circ F_t(y) = F_{t+s}(y)$ tends to 1 as $s \to \infty$. Hence $F_t(y) \geq \tilde{\mathbf{e}}$ on the event $\{y > \mathbf{e}\}$. Similarly, we have $F_t(y) \leq \tilde{\mathbf{e}}$ on the event $\{y < \mathbf{e}\}$. From these considerations we conclude that $\tilde{\mathbf{e}} \in [F_t(\mathbf{e}-), F_t(\mathbf{e})]$. Since $\tilde{\mathbf{e}}$ is independent of F_t, we see that the conditional law of $\mathbf{e}$ given F_t (or equivalently given ρ_t) is that of $F_t^{-1}(U)$, where U is uniform on $[0, 1]$ and independent of F_t. It follows that

$$\mathbb{E}\left(\varphi(\rho_t(\{\mathbf{e}\})) \mid \rho_t\right) = \int \rho_t(dy)\, \varphi(\rho_t(\{y\}))$$

and

$$\mathbb{E}\left(\varphi(\rho_t(\{\mathbf{e}\}))\right) = \int Q_t^*(0, dy)\, \varphi(y),$$

where $Q_t^*(0, \cdot)$ is as in the statement of the theorem. The property $\mathbb{P}(\rho_t(\{\mathbf{e}\}) = 0 \mid \rho_t) = d_t$ also readily follows.

We then claim that the conditional distribution of $(\rho_{t+s}(\mathbf{e}))_{s \geq 0}$ given the event $\{\rho_t(\{\mathbf{e}\}) = 0\}$ is the law of $(\rho_s(\{\mathbf{e}\}))_{s \geq 0}$ (this makes sense only in the improper case, since otherwise one is conditioning on an event of zero probability). On the event $\{\rho_t(\{\mathbf{e}\}) = 0\}$, the function F_t is continuous at $\mathbf{e}$, and in the preceding notation we have $\tilde{\mathbf{e}} = F_t(\mathbf{e})$ and

$$\rho_{t+s}(\{\mathbf{e}\}) = F_{t+s}(\mathbf{e}) - F_{t+s}(\mathbf{e}-) = \tilde{F}_s(F_t(\mathbf{e})) - \tilde{F}_s(F_t(\mathbf{e})-) = \tilde{F}_s(\tilde{\mathbf{e}}) - \tilde{F}_s(\tilde{\mathbf{e}}-).$$

The claim now follows.

At this point we observe that the kernels Q_t^* satisfy the following continuity property: For every continuous function φ on $[0, 1]$,

$$\lim_{x \to 0} Q_t^* \varphi(x) = Q_t^* \varphi(0). \tag{22}$$

To get this property, we simply use the exchangeability of increments of F_t to write

$$
\begin{aligned}
Q_t^* \varphi(x) &= \frac{1}{x} \mathbb{E}\Big(F_t(x)\, \varphi(F_t(x))\Big) \\
&= (1-x)^{-1} \mathbb{E}\Big(\frac{1}{x} \int_0^{1-x} da\, (F_t(a+x) - F_t(a))\varphi(F_t(a+x) - F_t(a))\Big) \\
&= (1-x)^{-1} \mathbb{E}\Big(\frac{1}{x} \int_0^{1-x} da \int_a^{a+x} \rho_t(dy)\, \varphi(F_t(a+x) - F_t(a))\Big) \\
&= (1-x)^{-1} \mathbb{E}\Big(\int \rho_t(dy) \frac{1}{x} \int_{(y-x)^+}^{y \wedge (1-x)} da\, \varphi(F_t(a+x) - F_t(a))\Big)
\end{aligned}
$$

and it is now clear that the latter quantity converges to

$$\mathbb{E}\Big(\int \rho_t(dy)\, \varphi(\rho_t(\{y\}))\Big) = Q_t^* \varphi(0)$$

24

as $x \to 0$.

We now prove by induction on p that for every $0 < t_1 < \cdots < t_p$, and every nonnegative measurable functions $\varphi_1, \ldots, \varphi_p$ on $[0, 1]$, one has

$$\mathbb{E}\Big(\varphi_1(\rho_{t_1}(\{\mathbf{e}\})) \cdots \varphi_p(\rho_{t_p}(\{\mathbf{e}\}))\Big)$$
$$= \int Q_{t_1}^*(0, dz_1) Q_{t_2-t_1}^*(z_1, dz_2) \ldots Q_{t_p-t_{p-1}}^*(z_{p-1}, dz_p) \varphi(z_1) \cdots \varphi(z_p). \quad (23)$$

The case $p = 1$ has already been established. Let $p \geq 2$ and suppose that the result holds at order $p-1$. Assume for the moment that $\varphi_1, \ldots, \varphi_p$ are continuous and that φ_1 vanishes on $[0, \varepsilon]$ for some $\varepsilon > 0$. Now observe from Proposition 1 that, for every $y > 0$, the conditional law of $(F_t(y), t \geq 0)$ given $\lim_{t \to \infty} F_t(y) = 1$ can be expressed as the h-transform of the unconditional law, for $h(y) = y$. Next, pick some integer $n \geq 1$ and for $k = 0, \ldots, n-1$, consider the processes

$$F_t((k+1)/n) - F_t(k/n), \qquad t \geq 0.$$

It follows from the exchangeability of the increments of bridges and the flow property that the distribution of each of these processes does not depend on k. The same thus holds true if we condition these processes to have limit 1 as $t \to \infty$, or equivalently by $\mathbf{e} \in]k/n, (k+1)/n]$. These conditional distributions are therefore all the same as that of $(F_t(1/n), t \geq 0)$ given $\lim_{t \to \infty} F_t(1/n) = 1$. By expressing $\rho_t(\{\mathbf{e}\})$ as the limit of $F_t([n\mathbf{e}+1]/n) - F_t([n\mathbf{e}]/n)$ when $n \to \infty$, we now get

$$\mathbb{E}\Big(\varphi_1(\rho_{t_1}(\{\mathbf{e}\})) \cdots \varphi_p(\rho_{t_p}(\{\mathbf{e}\}))\Big)$$
$$= \lim_{n \to \infty} \sum_{k=1}^{n} \mathbb{E}\Big(\varphi_1(F_{t_1}(\tfrac{k+1}{n}) - F_{t_1}(\tfrac{k}{n})) \ldots \varphi_p(F_{t_p}(\tfrac{k+1}{n}) - F_{t_p}(\tfrac{k}{n})) \, \mathbf{1}\{\lim_{t \to \infty} F_t(\tfrac{k+1}{n}) - F_t(\tfrac{k}{n}) = 1\}\Big)$$
$$= \lim_{n \to \infty} n \, \mathbb{E}\Big(\varphi_1(F_{t_1}(\tfrac{1}{n})) \ldots \varphi_p(F_{t_p}(\tfrac{1}{n})) \, \mathbf{1}\{\lim_{t \to \infty} F_t(\tfrac{1}{n}) = 1\}\Big)$$
$$= \lim_{n \to \infty} n \, \mathbb{E}\Big(\varphi_1(F_{t_1}(\tfrac{1}{n})) F_{t_1}(\tfrac{1}{n}) \int Q_{t_2-t_1}^*(F_{t_1}(\tfrac{1}{n}), dz_2) \ldots Q_{t_p-t_{p-1}}^*(z_{p-1}, dz_p) \varphi(z_2) \cdots \varphi(z_p)\Big)$$
$$= \int Q_{t_1}^*(0, dz_1) Q_{t_2-t_1}^*(z_1, dz_2) \ldots Q_{t_p-t_{p-1}}^*(z_{p-1}, dz_p) \varphi(z_1) \cdots \varphi(z_p),$$

where the last equality follows from (22). By standard arguments, we thus obtain that (23) holds for any nonnegative measurable functions $\varphi_1, \ldots, \varphi_p$ such that $\varphi_1(0) = 0$. To complete the proof, it remains to consider the case when $\varphi_1 = \mathbf{1}_{\{0\}}$. In that case however, the desired result follows from the fact that the conditional law of $(\rho_{t_1+s}(\{\mathbf{e}\}))_{s \geq 0}$ knowing that $\{\rho_{t_1}(\{\mathbf{e}\}) = 0\}$ is the law of $(\rho_s(\{\mathbf{e}\}))_{s \geq 0}$, together with the induction hypothesis. This completes the proof of (23).

It follows from (23) that the process $(\rho_t(\{\mathbf{e}\}))_{t \geq 0}$ is Markovian with semigroup Q_t^*. The Feller property of the semigroup follows from (22) and the Feller property of Q_t. Finally, in the proper case we have seen that $\mathbb{P}(\rho_t(\{\mathbf{e}\}) = 0) = 0$ for every $t > 0$, so that 0 must be an instantaneous point. On the other hand, in the proper case, we have $\mathbb{P}(\rho_t(\{\mathbf{e}\}) = 0) = \mathbb{E}(d_t) > 0$, so that $\mathbf{e}$ is a holding point. $\qquad \square$

In the improper case, the process $\rho_t(\{\mathbf{e}\})$ must leave 0 by a jump, that is, if $T := \inf\{t \geq 0 : \rho_t(\{\mathbf{e}\}) > 0\}$, we have $\rho_T(\{\mathbf{e}\}) > 0$. It is easy to compute the distribution of this jump when

B is the flow associated with the Λ-coalescent. Then the improper case holds iff

$$\int x^{-1}\Lambda(dx) < \infty$$

(see [15], Theorem 8), and the law of $\rho_T(\{\mathbf{e}\})$ is given by

$$\mathbb{E}\big(\varphi(\rho_T(\{\mathbf{e}\}))\big) = \frac{\int \varphi(x)x^{-1}\Lambda(dx)}{\int x^{-1}\Lambda(dx)}.$$

To prove this, observe that for any continuous function φ on $[0,1]$ such that $\varphi(0) = 0$, we have

$$\mathbb{E}\big(\varphi(\rho_T(\{\mathbf{e}\}))\big) = \lim_{t\to 0}\frac{1}{\theta t}Q_t^*\varphi(0) = \lim_{t\to 0}\frac{1}{\theta t}\mathbb{E}\Big(\int \rho_t(dy)\,\varphi(\rho_t(\{y\}))\Big),$$

where θ is the parameter of the exponential time T. Consider the special case $\varphi(r) = r^p$ for $p \geq 1$. It follows from the duality relation (18) that

$$\mathbb{E}\Big(\int \rho_t(dy)\,\rho_t(\{y\})^p\Big) = \mathbb{P}(|\Pi_t^{p+1}| = 1).$$

On the other hand, from the form of the rates for the Λ-coalescent, we have

$$\lim_{t\to 0}\frac{1}{t}\mathbb{P}(|\Pi_t^{p+1}| = 1) = \beta_{p+1,p+1} = \int x^{p-1}\Lambda(dx).$$

The stated result now follows.

Remark. In the case of the flow associated with the Λ-coalescent, the process $\rho_t(\{\mathbf{e}\})$ has the same one-dimensional marginals as the process of the frequency at time t of the block containing 1 in the standard Λ-coalescent (see Proposition 30 in [15]). Note however that the two processes are quite different, since the latter has nondecreasing paths, which is not the case for $\rho_t(\{\mathbf{e}\})$.

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