1 STAT 205B homework 7 solutions

Problem 8.1.3. (Durrett). For $Z \sim N(0,1)$, $E(Z^2-1)^2 = 2$. Now, $\Delta_{m,n} \sim N(0, 2^{-n}t)$ and for a fixed n, $\Delta_{m,n}, m \leq 2^n$ are i.i.d., hence

$$E(\sum_{m\leq n} \Delta_{m,n}^2 - t) = E(\sum(\Delta_{m,n}^2 - 2^{-n}t)) = \sum_{m\leq 2^n} E(\Delta_{m,n}^2 - 2^{-n}t)^2 = 2^{-2n}t^2 2^n \cdot 2 = t^2 2^{1-n}.$$

Now, for any $\epsilon > 0$,

$$P(|\sum_{m\leq n} \Delta_{m,n}^2 - t| > \epsilon) \leq \epsilon^{-2} E(\sum_{m\leq n} \Delta_{m,n}^2 - t) = \epsilon^{-2} t^2 2^{1-n},$$

hence $\sum_n P(|\sum_{m\leq n} \Delta_{m,n}^2 - t| > \epsilon) < \infty$, thus by Borel-Cantelli Lemma, $\sum_{m\leq n} \Delta_{m,n}^2 \to t$ a.s.

Problem 8.2.1 (Durrett). By Markov property,

$$\begin{aligned} P_x(R > 1+t) = P_x(T_0 \circ \theta_1 > t) = E_x[E_x[1(T_0 > t) \circ \theta_1 | \mathcal{F}_1^+]] &= E_x[E_{B_1}(1(T_0 > t))] \\ &= \int p_1(x,y) P_y(T_0 > t) dy. \end{aligned}$$

Problem 8.2.2 (Durrett). $\{L \leq t\} = \{T_0 \circ \theta_t > 1-t\}$, and hence by Markov property

$$P_0(L \leq t) = E_0[E_0[1(T_0 > 1-t) \circ \theta_t | \mathcal{F}_t^+] = E_0[P_{B_t}(T_0 > t)] = \int p_t(0,y) P_y(T_0 > 1-t) dy.$$

Problem 8.2.3 (Durrett). Theorem 8.2.5 implies that for any neighbourhood $(0, t_0)$, almost surely there exists a point $0 < t_1 < t_0$ such that $B_{t_1} = 0$. Again Theorem 8.2.4 gives that almost surely there exists $t_2 \in (0, t_1)$ such that $B_{t_2} > 0$. Since B_t has continuous paths almost surely, B_t attains a maxima at $t_3 \in [0, t_1]$, and as $B_{t_2} > 0$ and $B_0 = B_{t_1} = 0$, the maxima is attained at $t_3 \in (0, t_1)$. The given problem now follows by conditioning on $\mathcal{F}_a^+$ and applying the Markov property.

Problem 8.5.1 (Durrett). By Theorem 8.5.6 $\exp(\theta B_t - 2^{-1}\theta^2 t)$ is a martingale. Thus,

$$1 = E_0 \exp(\theta B(T \wedge t) - \theta^2 (T \wedge t)/2).$$

By Bounded convergence theorem, as $t \to \infty$,

$$\begin{aligned} 1 = E_0 \exp(\theta B_T - \theta^2 T/2) = P_0(B_T = a) E_0(\exp(\theta a - \theta^2 T/2) | B_T = a) + \\ P_0(B_T = -a) E_0(\exp(-\theta a - \theta^2 T/2) | B_T = -a). \end{aligned}$$

By symmetry, $E_0(\exp(-\theta^2 T/2) | B_T = a) = E_0(\exp(-\theta^2 T/2) | B_T = -a) = E_0(\exp(-\theta^2 T/2))$. Thus putting $\theta^2/2 = \lambda$, the result follows.

Problem 8.5.3 (Durrett). Observe that $\sigma = T_a \wedge T_b$. Also

$$E_x[\exp(-\lambda T_a)1(T_b < T_a)] = E_x[\exp(-\lambda\sigma)\exp(-\lambda(T_a - T_b))1(T_b < T_a)].$$

Conditioning on $\mathcal{F}_{T_b}$ and noting that $\exp(-\lambda\sigma)1(T_b < T_a) = \exp(-\lambda T_b)1(T_b < T_a)$ is $\mathcal{F}_{T_b}$ measurable,

$$\begin{aligned} E_x[\exp(-\lambda\sigma)\exp(-\lambda(T_a - T_b))1(T_b < T_a)] &= E_x[E_x[\exp(-\lambda\sigma)\exp(-\lambda(T_a \circ \theta_{T_b}))1(T_b < T_a)|\mathcal{F}_{T_b}]] \\ = E_x[\exp(-\lambda\sigma)1(T_b < T_a)E_{B_{T_b}}[\exp(-\lambda T_a)]] &= E_x[\exp(-\lambda\sigma)1(T_b < T_a)E_b[\exp(-\lambda T_a)]] \\ &= E_x[\exp(-\lambda\sigma)1(T_b < T_a)]E_b[\exp(-\lambda T_a)]. \end{aligned}$$

The rest follows directly from here.

Problem 8.5.4 (Durrett).
Since $B_t^4 - 6tB_t^2 + 3t^2$ is a martingale,

$$E[B_{T\wedge t}^4 - 6(T \wedge t)B_{T\wedge t}^2 + 3(T \wedge t)^2] = 0.$$

Thus, applying dominated convergence theorem for the first two terms and monotone convergence theorem for the third term, we have by letting $t \to \infty$,

$$E[B_T^4 - 6TB_T^2 + 3T^2] = 0.$$

Thus by Cauchy-Schwarz inequality,

$$3ET^2 \le E(B_T^4) + 3ET^2 = 6ETB_T^2 \le 6\sqrt{ET^2EB_T^4}.$$

Thus, $ET^2 \le 4E(B_T^4)$.

Problem 8.5.6 (Durrett). If $Z \sim N(0,1)$, then for $a > 1, b \in \mathbb{R}$, by standard computation,

$$E\exp((Z+b)^2/2a) = \sqrt{a/a-1}\exp(b^2/2(a-1)).$$

Thus, conditioning $(1+t)^{-1/2}\exp(B_t^2/2(1+t))$ by $\mathcal{F}_s$ and writing $B_t = B_t - B_s + B_s$ and using the above display, we get that

$$E[(1+t)^{-1/2}\exp(B_t^2/2(1+t))|\mathcal{F}_s] = (1+s)^{-1/2}\exp(B_s^2/2(1+s)).$$

Now, since $(1+t)^{-1/2}\exp(B_t^2/2(1+t))$ is a positive martingale, hence

$$\exp(\ln(1+t)(B_t^2/2(1+t)\ln(1+t) - 1/2)) = (1+t)^{-1/2}\exp(B_t^2/2(1+t)) \to X \quad a.s.$$

as $t \to \infty$ ans $X < \infty$ a.s.. From here it follows that

$$\limsup B_t/(1+t)\ln(1+t)^{1/2} \le 1 \quad a.s.$$