1 STAT 205B homework 7 solutions

Problem 8.1.3. (Durrett). For $Z \sim N(0, 1)$, $E(Z^2 - 1)^2 = 2$. Now, $\Delta_{m,n} \sim N(0, 2^{-n}t)$ and for a fixed $n$, $\Delta_{m,n}, m \leq 2^n$ are i.i.d., hence

$$E(\sum_{m \leq n} \Delta_{m,n}^2 - t) = E(\sum_{m \leq 2^n} (\Delta_{m,n}^2 - 2^{-n}t)) = \sum_{m \leq 2^n} E(\Delta_{m,n}^2 - 2^{-n}t)^2 = 2^{-2n}t^2 2^n \cdot 2 = t^2 2^{1-n}.$$

Now, for any $\epsilon > 0$,

$$P(|\sum_{m \leq n} \Delta_{m,n}^2 - t| > \epsilon) \leq \epsilon^{-2} E(\sum_{m \leq n} \Delta_{m,n}^2 - t) = \epsilon^{-2} t^2 2^{1-n},$$

hence $\sum_{m \leq n} P(|\sum_{m \leq n} \Delta_{m,n}^2 - t| > \epsilon) < \infty$, thus by Borel-Cantelli Lemma, $\sum_{m \leq n} \Delta_{m,n}^2 \to t$ a.s.

Problem 8.2.1 (Durrett). By Markov property,

$$P_x(R > 1 + t) = P_x(T_0 \circ \theta_t > t) = E_x[E_x[1(T_0 > t) \circ \theta_t | \mathcal{F}^+_t]] = E_x[E_{B_t}(1(T_0 > t))]
\quad = \int p_1(x,y)P_y(T_0 > t)dy.$$

Problem 8.2.2 (Durrett). $\{L \leq t\} = \{T_0 \circ \theta_t > 1 - t\}$, and hence by Markov property

$$P_0(L \leq t) = E_0[1(T_0 > 1 - t) \circ \theta_t] = E_0[P_{B_t}(T_0 > t)] = \int p_t(0,y)P_y(T_0 > 1 - t)dy.$$

Problem 8.2.3 (Durrett). Theorem 8.2.5 implies that for any neighbourhood $(0, t_0)$, almost surely there exists a point $0 < t_1 < t_0$ such that $B_{t_1} = 0$. Again Theorem 8.2.4 gives that almost surely there exists $t_2 \in (0, t_1)$ such that $B_{t_2} > 0$. Since $B_t$ has continuous paths almost surely, $B_t$ attains a maxima at $t_3 \in [0, t_1]$, and as $B_{t_2} > 0$ and $B_0 = B_{t_1} = 0$, the maxima is attained at $t_3 \in (0, t_1)$. The given problem now follows by conditioning on $\mathcal{F}^+_a$ and applying the Markov property.

Problem 8.5.1 (Durrett). By Theorem 8.5.6 $\exp(\theta B_t - 2^{-1} \theta^2 t)$ is a martingale. Thus,

$$1 = E_0 \exp(\theta B(t \wedge t) - \theta^2 (T \wedge t)/2).$$

By Bounded convergence theorem, as $t \to \infty$,

$$1 = E_0 \exp(\theta B_T - \theta^2 T/2) = P_0(B_T = a)E_0(\exp(\theta a - \theta^2 T/2)|B_T = a) + P_0(B_T = -a)E_0(\exp(-\theta a - \theta^2 T/2)|B_T = -a).$$

By symmetry, $E_0(\exp(-\theta^2 T/2)|B_T = a) = E_0(\exp(-\theta^2 T/2)|B_T = -a) = E_0(\exp(-\theta^2 T/2)).$

Thus putting $\theta^2/2 = \lambda$, the result follows.
Problem 8.5.3 (Durrett). Observe that $\sigma = T_a \wedge T_b$. Also
\[
E_x[\exp(-\lambda T_a)1(T_b < T_a)] = E_x[\exp(-\lambda)\exp(-\lambda(T_a - T_b))1(T_b < T_a)].
\]
Conditioning on $\mathcal{F}_{T_b}$ and noting that $\exp(-\lambda T_a)1(T_b < T_a) = \exp(-\lambda T_b)1(T_b < T_a)$ is $\mathcal{F}_{T_b}$ measurable,
\[
E_x[\exp(-\lambda T_a)1(T_b < T_a)] = E_x[E_x[\exp(-\lambda T_a)1(T_b < T_a)|\mathcal{F}_{T_b}]]
= E_x[\exp(-\lambda T_b)1(T_b < T_a)E_{T_a}[\exp(-\lambda T_a)]] = E_x[\exp(-\lambda T_a)1(T_b < T_a)]E_{T_a}[\exp(-\lambda T_a)].
\]
The rest follows directly from here.

Problem 8.5.4 (Durrett).
Since $B_t^4 - 6tB_t^2 + 3t^2$ is a martingale,
\[
E[B_t^4 - 6(T \wedge t)B_t^2 + 3(T \wedge t)^2] = 0.
\]
Thus, applying dominated convergence theorem for the first two terms and monotone convergence theorem for the third term, we have by letting $t \to \infty$,
\[
E[B_t^4 - 6T B_t^2 + 3T^2] = 0.
\]
Thus by Cauchy-Schwarz inequality,
\[
3ET^2 \leq E(B_t^4) + 3ET^2 = 6ET B_t^2 \leq 6\sqrt{ET^2 E B_t^4}.
\]
Thus, $ET^2 \leq 4E(B_t^4)$.

Problem 8.5.6 (Durrett). If $Z \sim N(0,1)$, then for $a > 1, b \in \mathbb{R}$, by standard computation,
\[
E \exp((Z + b)^2 / 2a) = \sqrt{a/a - 1} \exp(b^2 / 2(a - 1)).
\]
Thus, conditioning $(1 + t)^{-1/2} \exp(B_t^2 / 2(1 + t))$ by $\mathcal{F}_s$ and writing $B_t = B_t - B_s + B_s$ and using the above display, we get that
\[
E[(1 + t)^{-1/2} \exp(B_t^2 / 2(1 + t))]|\mathcal{F}_s] = (1 + s)^{-1/2} \exp(B_s^2 / 2(1 + s)).
\]
Now, since $(1 + t)^{-1/2} \exp(B_t^2 / 2(1 + t))$ is a positive martingale, hence
\[
\exp(\ln(1 + t)(B_t^2 / 2(1 + t) \ln(1 + t) - 1/2)) = (1 + t)^{-1/2} \exp(B_t^2 / 2(1 + t)) \to X \quad a.s.
\]
as $t \to \infty$ ans $X < \infty$ a.s. From here it follows that
\[
\limsup B_t/(1 + t) \ln(1 + t)^{1/2} \leq 1 \quad a.s.
\]