1 STAT 205B homework 6 solutions

Problem 7.1.2 (Durrett). (i) $\phi^{-1}(B) = \cup_{n=1}^{\infty} \phi^{-n}(A) \subseteq B$
(ii) Clearly $\phi^{-1}(C) \supseteq C$. Also as $\phi^{-1}(B) \subseteq B$, one has $\phi^{-1}(C) \subseteq C$.
(iii) Define $B$ and $C$ as above, then $C$ is strictly invariant. Since $\phi$ is measure preserving, $P(\phi^{-n}(A) \Delta \phi^{-1}(A)) = P(A \Delta \phi^{-1}(A)) = 0$, where the last inequality follows because $A$ is almost invariant, and hence

$$P(\phi^{-n}(A)) \leq \sum_{k=0}^{n-1} P(\phi^{-k}(A) \Delta \phi^{-1}(A)) = 0.$$ 

Thus $P(A \Delta B) \leq \sum_{n=0}^{\infty} P(A \Delta \phi^{-n}(A)) = 0$.
Again, since $B \setminus \phi^{-1}(B) \subseteq A \setminus \phi^{-1}(A)$, using similar argument as above, it follows that $P(B \Delta C) = 0$. Hence, $P(A \Delta C) = 0$.

Problem 7.1.5 (Durrett). It is enough to show that $\mu(\phi^{-1}([a,b])) = \mu([a,b])$ for all $0 \leq a < b \leq 1$. Now,

$$\mu(\phi^{-1}([a,b])) = \sum_{n=1}^{\infty} \mu([(n+b)^{-1}, (n+a)^{-1}])$$

$$= (\ln 2)^{-1} \sum_{n=1}^{\infty} [\ln(n+a+1) - \ln(n+b+1) - (\ln(n+a) - \ln(n+b))]$$

$$= \frac{\ln(1+b) - \ln(1+a)}{\ln 2} = \mu([a,b]).$$

Problem 7.1.6 (Durrett).
Stationarity.

$$P(Z_1, Z_2, \ldots \in A) = \sum_{j=1}^{n} P(Y_{\nu+1}, Y_{\nu+2}, \ldots \in A | \nu = j) P(\nu = j) = \frac{1}{n} \sum_{j=1}^{n} P(Y_{j+1}, Y_{j+2}, \ldots \in A)$$

so in order to prove that $Z_1, Z_2, \ldots \overset{d}{=} Z_2, Z_3, \ldots$, it is enough to show that

$$\sum_{j=1}^{n} P(Y_{j+1}, Y_{j+2}, \ldots \in A) = \sum_{j=1}^{n} P(Y_{j+2}, Y_{j+3}, \ldots \in A).$$

This holds, since $P(Y_2, Y_3, \ldots \in A) = P(Y_{n+2}, Y_{n+3}, \ldots \in A)$ as the $n$-blocks of $\{Y_n\}$ are iid.

Ergodicity. Let $B$ be a shift-invariant event so that $B = \{Z_1, Z_2, \ldots \in A\}$ for some set $A$. Using shift-invariance a few times we get that for all $m > n$ and possible values $j$ of $\nu$:

$$B = \{Z_{m-j}, Z_{m-j+1}, \ldots \in A\}$$

Intersecting everything with $\{\nu = j\}$ we get:

$$\{\nu = j\} \cap B = \{\nu = j\} \cap \{Y_{m}, Y_{m+1}, \ldots \in A\}$$
Taking union over all \( j \):

\[ B = \{ Y_m, Y_{m+1}, \ldots \in A \} \]

Since \( m \) can be arbitrarily large, \( B \) has to be in the tail \( \sigma \)-field of \( Y \), which is trivial by Kolmogorov’s 0-1 law.

**Problem 7.3.1 (Durrett).**

\[
ER_n = 1 + \sum_{m=1}^{n-1} E(\text{the last visit to } S_m \text{ up to time } n \text{ occurs at time } m) \\
= 1 + \sum_{m=1}^{n-1} P(X_{m+1} + \ldots + X_j \neq 0 \text{ for } j \in [m+1,n]) \quad \text{by stationarity:} \\
= 1 + \sum_{m=1}^{n-1} g_{n-m} = \sum_{m=1}^{n} g_{m-1}.
\]

**Problem 7.3.3 (Durrett).**

First note that by Kolmogorov’s Extension Theorem (more precisely, see Theorem (6.1.2)) we can embed \( \{X_i\} \) in a two-sided stationary sequence. By Fubini:

\[
E(\sum_{m=1}^{T_1} 1(X_m \in B, X_0 \in A)) = \sum_{m=1}^{\infty} P(X_m \in B, X_0 \in A, T_1 \geq m) \\
= \sum_{m=1}^{\infty} P(X_0 \in A, X_1 \not\in A, \ldots, X_{m-1} \not\in A, X_m \in B) \\
= \sum_{m=1}^{\infty} P(X_{-m} \in A, X_{-m+1} \not\in A, \ldots, X_{-1} \not\in A, X_0 \in B) = P(X_0 \in B)
\]

Here we used that \( X_n \in A \) for some \( n < 0 \) a.s; this is shown in the proof of Theorem 7.3.3.

**Problem 7.3.4 (Durrett).**

By Theorem 7.3.3., \( \bar{E}(T_1) = 1/P(X_0 = 1) \), hence

\[
\bar{P}(T_1 \geq n)/\bar{E}(T_1) = P(T_1 \geq n|X_0 = 1)P(X_0 = 1) = P(X_0 = 1, T_1 \geq n).
\]

The fact that this is equal to \( P(T_1 = n) \) was proved in Problem 9. (a).