1 STAT 205B homework 6 solutions

Problem 7.1.2 (Durrett). (i) $\phi^{-1}(B) = \cup_{n=1}^{\infty}\phi^{-n}(A) \subseteq B$
(ii) Clearly $\phi^{-1}(C) \supseteq C$. Also as $\phi^{-1}(B) \subseteq B$, one has $\phi^{-1}(C) \subseteq C$.
(iii) Define B and C as above, then C is strictly invariant. Since ϕ is measure preserving, $P(\phi^{-n}(A)\Delta\phi^{-n-1}(A)) = P(A\Delta\phi^{-1}(A)) = 0$, where the last inequality follows because A is almost invariant, and hence

$$P(A\Delta\phi^{-n}(A)) \leq \sum_{k=0}^{n-1} P(\phi^{-k}(A)\Delta\phi^{-k-1}(A)) = 0\,.$$

Thus $P(A\Delta B) \leq \sum_{n=0}^{\infty} P(A\Delta\phi^{-n}(A)) = 0\,.$

Again, since $B \setminus \phi^{-1}(B) \subseteq A \setminus \phi^{-1}(A)$, using similar argument as above, it follows that $P(B\Delta C) = 0$. Hence, $P(A\Delta C) = 0$.

Problem 7.1.5 (Durrett). It is enough to show that $\mu(\phi^{-1}([a,b])) = \mu([a,b])$ for all $0 \leq a \leq b \leq 1$. Now,

$$\begin{aligned}
\mu(\phi^{-1}([a,b])) &= \sum_{n=1}^{\infty} \mu([(n+b)^{-1},(n+a)^{-1}]) \\
&= (\ln 2)^{-1} \sum_{n=1}^{\infty} [\ln(n+a+1) - \ln(n+b+1) - (\ln(n+a) - \ln(n+b))] \\
&= \frac{\ln(1+b) - \ln(1+a)}{\ln 2} = \mu([a,b])\,.
\end{aligned}$$

Problem 7.1.6 (Durrett).

Stationarity.

$$P(Z_1, Z_2, ... \in A) = \sum_{j=1}^{n} P(Y_{\nu+1}, Y_{\nu+2}, ... \in A | \nu = j) P(\nu = j) = \frac{1}{n}\sum_{j=1}^{n} P(Y_{j+1}, Y_{j+2}, ... \in A)$$

so in order to prove that $Z_1, Z_2, ... \stackrel{d}{=} Z_2, Z_3...$, it is enough to show that

$$\sum_{j=1}^{n} P(Y_{j+1}, Y_{j+2}, ... \in A) = \sum_{j=1}^{n} P(Y_{j+2}, Y_{j+3}, ... \in A).$$

This holds, since $P(Y_2, Y_3, ... \in A) = P(Y_{n+2}, Y_{n+3}, ... \in A)$ as the n-blocks of $\{Y_n\}$ are iid.

Ergodicity. Let B be a shift-invariant event so that $B = \{Z_1, Z_2, .. \in A\}$ for some set A. Using shift-invariance a few times we get that for all $m > n$ and possible values j of ν:

$$B = \{Z_{m-j}, Z_{m-j+1}, ... \in A\}$$

Intersecting everything with $\{\nu = j\}$ we get :

$$\{\nu = j\} \cap B = \{\nu = j\} \cap \{Y_m, Y_{m+1}, ... \in A\}$$

Taking union over all j:

$$B = \{Y_m, Y_{m+1}, ... \in A\}$$

Since m can be arbitrarily large, B has to be in the tail σ-field of Y, which is trivial by Kolmogorov's 0-1 law.

Problem 7.3.1 (Durrett).

$$\begin{aligned} ER_n &= 1 + \sum_{m=1}^{n-1} E\mathbf{1}(\text{the last visit to } S_m \text{ up to time } n \text{ occurs at time } m) \\ &= 1 + \sum_{m=1}^{n-1} P(X_{m+1} + ... + X_j \neq 0 \text{ for } j \in [m+1, n]) \quad \text{by stationarity:} \\ &= 1 + \sum_{m=1}^{n-1} g_{n-m} = \sum_{m=1}^{n} g_{m-1}. \end{aligned}$$

Problem 7.3.3 (Durrett).

First note that by Kolmogorov's Extension Theorem (more precisely, see Theorem (6.1.2)) we can embed $\{X_i\}$ in a two-sided stationary sequence. By Fubini:

$$\begin{aligned} E(\sum_{m=1}^{T_1} \mathbf{1}(X_m \in B, X_0 \in A)) &= \sum_{m=1}^{\infty} P(X_m \in B, X_0 \in A, T_1 \geq m) \\ &= \sum_{m=1}^{\infty} P(X_0 \in A, X_1 \notin A, ..., X_{m-1} \notin A, X_m \in B) \\ &= \sum_{m=1}^{\infty} P(X_{-m} \in A, X_{-m+1} \notin A, ..., X_{-1} \notin A, X_0 \in B) = P(X_0 \in B) \end{aligned}$$

Here we used that $X_n \in A$ for some $n < 0$ a.s; this is shown in the proof of Theorem 7.3.3.

Problem 7.3.4 (Durrett).

By Theorem 7.3.3., $\bar{E}(T_1) = 1/P(X_0 = 1)$, hence

$$\bar{P}(T_1 \geq n)/\bar{E}(T_1) = P(T_1 \geq n | X_0 = 1)P(X_0 = 1) = P(X_0 = 1, T_1 \geq n).$$

The fact that this is equal to $P(T_1 = n)$ was proved in Problem 9. (a).