1 STAT 205B homework solutions; HW 2

Problem 6.2.7 (Durrett).

Observe that

$$P(X_{n+1} = x_n + 1 | X_k = x_k, 1 \le k \le n) = P(\xi_{n+1} \notin \{\xi_1, ..\xi_n\} | X_k = x_k, 1 \le k \le n)$$

$$= \frac{N - x_n}{N} \quad \textit{by independence}$$

For $x_n = x_{n+1}$ we get $P(X_{n+1} = x_n | X_k = x_k, 1 \le k \le n) = \frac{x_n}{N}$ and for all other cases 0. We see that $P(X_{n+1} = x_n + 1 | \sigma(X_1, ..X_n))$ is $\sigma(X_n)$-measurable and thus $P(X_{n+1} = x_n + 1 | \sigma(X_1, ..X_n)) = P(X_{n+1} = x_n + 1 | \sigma(X_n))$, so X is a Markov chain.

Problem 6.2.8 (Durrett).

Consider the sequence $(X_1, X_2, X_3) = (1, 1, 1)$. This can happen in two ways: $(S_1, S_2, S_3) = (1, 0, -1)$, or $(S_1, S_2, S_3) = (1, 0, 1)$. Thus, $P(X_4 = 2 | X_1 = 1, X_2 = 1, X_3 = 1) = .25$. Now consider the sequence $(X_1, X_2, X_3) = (0, 0, 1)$. This can only happen if $(S_1, S_2, S_3) = (-1, 0, 1)$. Thus, $P(X_4 = 2 | X_1 = 0, X_2 = 0, X_3 = 1) = .5$. Thus, X is not a Markov chain.

Problem 6.2.9 (Durrett).

Let $p(s_n) = \frac{E\theta^{\frac{s_n+n+2}{2}}(1-\theta)^{\frac{n-s_n}{2}}}{E\theta^{\frac{s_n+n}{2}}(1-\theta)^{\frac{n-s_n}{2}}} = \frac{s_n+n+2}{2(n+2)}$. We will show $P(X_{n+1} = 1 | S_1, ..S_n) = p(S_n)$. Then all the assertions will follow.

$$\int_{[S_1=s_1,..S_n=s_n]} p(S_n) dP = p(s_n) P(S_1 = s_1, ..S_n = s_n)$$

$$= p(s_n) EP(U_k \le \theta \ for \ k \in A, U_k > \theta \ for \ k \in B | \theta)$$

where $A = \{k : s_k - s_{k-1} = 1\}, B = \{k : s_k - s_{k-1} = -1\}$

$$= p(s_n) E\theta^{|A|}(1-\theta)^{|B|} \quad \textit{by independence}$$

$$= p(s_n) E\theta^{\frac{s_n+n}{2}}(1-\theta)^{\frac{n-s_n}{2}} = E\theta^{\frac{s_n+n+2}{2}}(1-\theta)^{\frac{n-s_n}{2}} = E\theta^{|A|+1}(1-\theta)^{|B|}$$

$$= EP(U_k \le \theta \ for \ k \in A \cup \{n+1\}, U_k > \theta \ for \ k \in B | \theta)$$

$$= P(S_1 = s_1, ..S_n = s_n, X_{n+1} = 1) = \int_{[S_1=s_1,..S_n=s_n]} 1_{[X_{n+1}=1]} dP$$

Problem 6.3.10 (Durrett).

(i) $x \notin A$ gives $\tau_A \ge 1$ and so

$$E_x(\tau_A | \mathcal{F}_1) = E_{X_1}(\tau_A + 1)$$

as $\tau_A \ge 1$ gives $\tau_A = 1 + \tau_A \circ \theta_1$. Thus

$$g(x) = E_x \tau_A = E_x(E_x(\tau_A | \mathcal{F}_1)) = 1 + E_x(E_{X_1}(\tau_A)) = 1 + E_x g(X_1) = 1 + \sum_y p(x, y) g(y).$$

(ii)

$$E[g(X((n+1)\wedge\tau_A)) + ((n+1)\wedge\tau_A)|\mathcal{F}_n] = \mathbf{1}\{\tau_A \le n\}(g(X(\tau_A)) + \tau_A) + \mathbf{1}\{\tau_A > n\}(g(X(n)) - 1 + n +$$
$$= g(X(n\wedge\tau_A)) + (n\wedge\tau_A)$$

using (*)

(iii) First, observe that $P_x(\tau_A < \infty) > 0$ gives the existence of an n(x) with $p^{n(x)}(x, A) > 0$, so as $S - A$ is finite we get for some $N < \infty, \epsilon > 0$ that

$$\begin{aligned} E_x\tau_A &= \sum_{k=0}^{\infty} P_x(\tau_A > k) \text{ as } \tau_A \in \{0, 1, 2, ..\} \\ &\le \sum_{k=0}^{\infty} NP_x(\tau_A > kN) \text{ as } P_x(\tau_A > m) \le P_x(\tau_A > n) \text{ if } m \ge n \\ &\le N\sum_{k=0}^{\infty}(1-\epsilon)^k < \infty \quad as\ \epsilon > 0. \end{aligned}$$

That is, $E_x\tau_A$ is a bounded function.

The conditional expectation of the above martingale given $X(0)$ is (using the $n = 0$ case) $g(X(0))$. On the other hand, since $S - A$ is finite, and g is 0 on A, g is a bounded function, and convergence of the martingale to $g(X(\tau_A)) + \tau_A = \tau_A$ gives that $E[\tau_A|X(0)] = g(X(0))$.

Problem 6.3.11 (Durrett).

Observe $p((V, H), (H, W)) = p((V, T), (T, W)) = 1/2$ for $V, W \in \{H, T\}$, $p(\cdot, \cdot) = 0$ else. Let $A = (H, H)$ and check that all conditions in 2.11. are satisfied. So E_xN_1 is the only solution of

$$g(T, H) = 1 + 1/2(g(H, H) + g(H, T)) = 1 + 1/2g(H, T)$$

max

$$g(T, T) = 1 + 1/2(g(T, T) + g(T, H))$$
$$g(H, T) = 1 + 1/2(g(T, T) + g(T, H))$$

with $g(H, H) = 0$.
Find $g(H, T) = g(T, T) = 2 + g(T, H) = 6$.
So $EN_1 = 4$.

Problem 6.3.12 (Durrett).

Let $A = \{0, N\}$ and use the result of 6.3.10. It is easy to check that $g(x) = x(N - x)$ solves the system of equations (*), and thus, $g(x) = E_x\tau_A$.

Problem 10.

Let X be a simple symmetric random walk on $\mathbb{Z}$ with $X_0 = 0$ and let $f(x) = x$ if x is odd, $f(x) = 0$ if x is even. Then $P(f(X_3) = 3|f(X_2) = 0) = P(f(X_3) = 3) = \frac{1}{8}$ while $P(f(X_3) = 3|f(X_2) = 0, f(X_1) = -1) = 0$. Thus, $(f(X_n))$ is not a Markov chain.

Problem 11.

Let $n_0(i,j) = \min\{n : P^n(i,j) > 0\}$. Then $n_0(i,j) < K$ because we can always avoid a cycle. Now

$$\max_{i\neq j} P_i(T_j > K-1) \leq \max_{i\neq j} P_i(T_j > n_0(i,j)) = \max_{i\neq j}(1 - p^{n_0(i,j)}(i,j))$$

$$\leq \max_{i\neq j}(1 - a^{n_0(i,j)}) \leq 1 - a^{K-1}$$

Let $P_{(-j)} = ((p(i,k) : i,k \neq j))$. Then

$$\| P_{(-j)}^{K-1} \| = \max_{i\neq j} \sum_{l\neq j} P_{(-j)}^{K-1}(i,l) = \max_{i\neq j} P_i(T_j > K-1) \leq 1 - a^{K-1}$$

and $\| P_{(-j)}^{m(K-1)} \| \leq (1 - a^{K-1})^m,\ m = 0, 1, 2, ...$

So

$$E_i T_j = \sum_{n=0}^{\infty} P_i(T_j > n) = \sum_{m=0}^{\infty} \sum_{n=m(K-1)}^{(m+1)(K-1)-1} P_i(T_j > n)$$

$$\leq \sum_{m=0}^{\infty} (K-1) P_i(T_j > m(K-1)) \leq \sum_{m=0}^{\infty} (K-1) \| P_{(-j)}^{m(K-1)} \|$$

$$\leq (K-1) \sum_{m=0}^{\infty} (1 - a^{K-1})^m = \frac{K-1}{a^{K-1}}$$

Further, $E_j T_j \leq 1 + \max_i E_i T_j$

So we can take $C(K,a) = 1 + \frac{K-1}{a^{K-1}}$.