1 STAT 205B homework solutions; HW 2

Problem 6.2.7 (Durrett).

Observe that

\[
P(X_{n+1} = x_n + 1|X_k = x_k, 1 \leq k \leq n) = P(\xi_{n+1} \notin \{\xi_1, \ldots, \xi_n\}|X_k = x_k, 1 \leq k \leq n)
\]

\[
= \frac{N - x_n}{N} \quad \text{by independence}
\]

For \(x_n = x_{n+1}\) we get \(P(X_{n+1} = x_n|X_k = x_k, 1 \leq k \leq n) = \frac{x_n}{N}\) and for all other cases 0. We see that \(P(X_{n+1} = x_n + 1|\sigma(X_1, \ldots, X_n)) = \sigma(X_n)\)-measurable and thus \(P(X_{n+1} = x_n + 1|\sigma(X_1, \ldots, X_n)) = P(X_{n+1} = x_n + 1|\sigma(X_n))\), so \(X\) is a Markov chain.

Problem 6.2.8 (Durrett).

Consider the sequence \((X_1, X_2, X_3) = (1, 1, 1)\). This can happen in two ways: \((S_1, S_2, S_3) = (1, 0, -1)\), or \((S_1, S_2, S_3) = (1, 0, 1)\). Thus, \(P(X_4 = 2|X_1 = 1, X_2 = 1, X_3 = 1) = .25\). Now consider the sequence \((X_1, X_2, X_3) = (0, 0, 1)\). This can only happen if \((S_1, S_2, S_3) = (-1, 0, 1)\). Thus, \(P(X_4 = 2|X_1 = 0, X_2 = 0, X_3 = 1) = .5\). Thus, \(X\) is not a Markov chain.

Problem 6.2.9 (Durrett).

Let \(p(s_n) = \frac{E\theta^{n+s_n}}{E\theta^{n+\frac{n+1}{2}(1-\theta)^\frac{n+s_n}{2}}(1-\theta)^\frac{n-s_n}{2}} = \frac{s_n + n^2 + 2}{2(n+2)}\). We will show \(P(X_{n+1} = 1|S_1, \ldots, S_n) = p(S_n)\). Then all the assertions will follow.

\[
\int_{[S_1=s_1, \ldots, S_n=s_n]} p(S_n)dP = p(s_n)P(S_1 = s_1, \ldots, S_n = s_n)
\]

\[
= p(s_n)EP(U_k \leq \theta \text{ for } k \in A, U_k > \theta \text{ for } k \in B|\theta)
\]

where \(A = \{k : s_k - s_{k-1} = 1\}\), \(B = \{k : s_k - s_{k-1} = -1\}\)

\[
= p(s_n)E\theta^{|A|}(1-\theta)^{|B|} \quad \text{by independence}
\]

\[
= p(s_n)E\theta^{\frac{n+s_n}{2}}(1-\theta)^{-\frac{n-s_n}{2}} = E\theta^{\frac{n+s_n+2}{2}}(1-\theta)^{\frac{n-s_n}{2}} = E\theta^{|A|+1}(1-\theta)^{|B|}
\]

\[
= EP(U_k \leq \theta \text{ for } k \in A \cup \{n+1\}, U_k > \theta \text{ for } k \in B|\theta)
\]

\[
P(S_1 = s_1, \ldots, S_n = s_n, X_{n+1} = 1) = \int_{[S_1=s_1, \ldots, S_n=s_n]} 1_{[X_{n+1}=1]}dP
\]

Problem 6.3.10 (Durrett).

(i) \(x \notin A\) gives \(\tau_A \geq 1\) and so

\[
E_x(\tau_A|\mathcal{F}_1) = E_{X_1}(\tau_A + 1)
\]

as \(\tau_A \geq 1\) gives \(\tau_A = 1 + \tau_A \circ \theta_1\). Thus

\[
g(x) = E_x\tau_A = E_x(E_x(\tau_A|\mathcal{F}_1)) = 1 + E_x(E_{X_1}(\tau_A)) = 1 + E_x g(X_1) = 1 + \sum_y p(x, y)g(y).
\]
(ii)

\[ E[g(X((n+1) \land \tau_A)) + ((n+1) \land \tau_A)|\mathcal{F}_n] = 1 \{\tau_A \leq n\}(g(X(\tau_A)) + \tau_A) + 1\{\tau_A > n\}(g(X(n)) - 1 + n + \tau_A) = g(X(n \land \tau_A)) + (n \land \tau_A) \]

using (*)

(iii) First, observe that \( P_x(\tau_A < \infty) > 0 \) gives the existence of an \( n(x) \) with \( p^n(x)(x, A) > 0 \), so as \( -S - A \) is finite we get for some \( N < \infty, \epsilon > 0 \) that

\[
E_x \tau_A = \sum_{k=0}^{\infty} P_x(\tau_A > k) \text{ as } \tau_A \in \{0, 1, 2, \ldots\}
\leq \sum_{k=0}^{\infty} NP_x(\tau_A > kN) \text{ as } P_x(\tau_A > m) \leq P_x(\tau_A > n) \text{ if } m \geq n
\leq N \sum_{k=0}^{\infty} (1 - \epsilon)^k < \infty \text{ as } \epsilon > 0.
\]

That is, \( E_x \tau_A \) is a bounded function.

The conditional expectation of the above martingale given \( X(0) \) is (using the \( n = 0 \) case) \( g(X(0)) \). On the other hand, since \( S - A \) is finite, and \( g \) is 0 on \( A \), \( g \) is a bounded function, and convergence of the martingale to \( g(X(\tau_A)) + \tau_A = \tau_A \) gives that \( E[\tau_A|X(0)] = g(X(0)) \).

**Problem 6.3.11 (Durrett).**

Observe \( p((V, H), (H, W)) = p((V, T), (T, W)) = 1/2 \) for \( V, W \in \{H, T\}, \ p(\cdot, \cdot) = 0 \) else.

Let \( A = (H, H) \) and check that all conditions in 2.11. are satisfied. So \( E_x N_1 \) is the only solution of

\[
g(T, H) = 1 + 1/2(g(H, H) + g(H, T)) = 1 + 1/2g(H, T)
\]

max

\[
g(T, T) = 1 + 1/2(g(T, T) + g(T, H))
g(H, T) = 1 + 1/2(g(T, T) + g(T, H))
\]

with \( g(H, H) = 0 \).

Find \( g(H, T) = g(T, T) = 2 + g(T, H) = 6 \).

So \( E N_1 = 4 \).

**Problem 6.3.12 (Durrett).**

Let \( A = \{0, N\} \) and use the result of 6.3.10. It is easy to check that \( g(x) = x(N - x) \) solves the system of equations (*), and thus, \( g(x) = E_x \tau_A \).

**Problem 10.**

Let \( X \) be a simple symmetric random walk on \( \mathbb{Z} \) with \( X_0 = 0 \) and let \( f(x) = x \) if \( x \) is odd, \( f(x) = 0 \) if \( x \) is even. Then \( P(f(X_3) = 3|f(X_2) = 0) = P(f(X_3) = 3) = \frac{1}{8} \) while \( P(f(X_3) = 3|f(X_2) = 0, f(X_1) = -1) = 0 \). Thus, \((f(X_n)) \) is not a Markov chain.
Problem 11.
Let \( n_0(i, j) = \min\{ n : P^n(i, j) > 0 \} \). Then \( n_0(i, j) < K \) because we can always avoid a cycle. Now

\[
\max_{i \neq j} P_i(T_j > K - 1) \leq \max_{i \neq j} P_i(T_j > n_0(i, j)) = \max_{i \neq j} (1 - p^{n_0(i, j)}(i, j))
\]

\[
\leq \max_{i \neq j} (1 - a^{n_0(i, j)}) \leq 1 - a^{K-1}
\]

Let \( P_{(-j)} = ((p(i, k) : i, k \neq j)) \). Then

\[
\| P_{K-1}^{(-j)} \| = \max_{i \neq j} \sum_{l \neq j} P_{K-1}^{(-j)}(i, l) = \max_{i \neq j} P_i(T_j > K - 1) \leq 1 - a^{K-1}
\]

and \( \| P_{m(K-1)}^{(-j)} \| \leq (1 - a^{K-1})^m, \ m = 0, 1, 2, \ldots \)

So

\[
E_i T_j = \sum_{n=0}^{\infty} P_i(T_j > n) = \sum_{m=0}^{\infty} \sum_{n=m(K-1)}^{(m+1)(K-1)-1} P_i(T_j > n)
\]

\[
\leq \sum_{m=0}^{\infty} (K - 1)P_i(T_j > m(K - 1)) \leq \sum_{m=0}^{\infty} (K - 1) \| P_{m(K-1)}^{(-j)} \|
\]

\[
\leq (K - 1) \sum_{m=0}^{\infty} (1 - a^{K-1})^m = \frac{K - 1}{a^{K-1}}
\]

Further, \( E_j T_j \leq 1 + \max_i E_i T_j \)
So we can take \( C(K, a) = 1 + \frac{K-1}{a^{K-1}} \).