Solution for HW 1

1. Assume that \( \pi, \nu, \mu \) are probability measures on \((S, \mathcal{S})\) satisfying \( \pi \ll \nu \ll \mu \). On one hand, \( \pi \ll \mu \) implies that \( \pi(A) = \int_A \frac{d\pi}{d\nu} d\nu \) for all \( A \in \mathcal{S} \). On the other hand, \( \nu \ll \mu \) gives that \( \int_S f d\nu = \int_S f \frac{d\nu}{d\mu} d\mu \) for all measurable \( f \geq 0 \). In particular, take \( f = 1_A \frac{d\pi}{d\nu} \) for some \( A \in \mathcal{S} \), we obtain

\[
\pi(A) = \int_A \frac{d\pi}{d\nu} d\nu = \int_A \frac{d\pi}{d\nu} \frac{d\nu}{d\mu} d\mu.
\]

Thus \( \pi \ll \mu \). By uniqueness of RN derivative, we have \( \frac{d\pi}{d\mu} = \frac{d\pi}{d\nu} \frac{d\nu}{d\mu} \) a.s.

2. Since \( S_2 \) is nice, we can reduce the problem to the case where \( S_2 = \mathbb{R} \). Denote \( \mathcal{A} := \{x; Q^*(x, B) = Q(x, B)\} \) for all \( B \in S_2 \) and \( \mathcal{A}_r := \{x; Q^*(x, (\infty, r)) = Q(x, (\infty, r))\} \) for all \( r \in \mathbb{Q} \). We have then \( \mathcal{A} = \cap_{r \in \mathbb{Q}} \mathcal{A}_r \) by appealing to \( \pi - \lambda \) argument. Note that for fixed \( B \in S_2 \), \( \mu(A \times B) = \int_A Q(x, B) d\mu_1(x) = \int_A Q^*(x, B) d\mu_1(x) \) for all \( A \in S_1 \). It is immediate that \( Q(x, B) = Q^*(x, B) \) \( \mu_1 \) a.e. In particular, \( \mu_1(\mathcal{A}_r) = \mu_1(\cup_{r \in \mathbb{Q}} \mathcal{A}_r^c) = 0. \) Therefore, \( \mu_1(x; Q^*(x, B) = Q(x, B)\) for all \( B \in S_2 = 1 \).

3. Denote \( X \) the random variable with the distribution function \( F \). The we have

\[
\int_{\mathbb{R}} F(x + c) - F(x) dx = \int_{\mathbb{R}} \mathbb{E} 1_{x < X \leq x+ c} dx \overset{(*)}{=} \mathbb{E} \int_{\mathbb{R}} 1_{X-c \leq x \leq X} dx = c,
\]

where \( (*) \) is due to Tonelli-Fubini theorem.

4. Fix \( a \in \mathbb{R} \). Observe that \( f(x, u) \leq a \iff g(u, x) := u - Q(x, (-\infty, a]) \leq 0 \). Remark that \( g \) is product measurable as the sum of two (product) measurable functions. Thus, the inverse distribution function is product measurable.

5. For \( 1 \leq i < j \leq 3 \), define the probability measures \( \mu_{ij} \) on \( \{0, 1\}^2 \) such that \( \mu_{ij}(\{0\} \times \{1\}) = \mu_{ij}(\{1\} \times \{0\}) = \frac{1}{2} \) and \( \mu_{ij}(\{0\} \times \{0\}) = \mu_{ij}(\{1\} \times \{1\}) = 0 \). Obviously, the consistency condition (2) is satisfied with \( \mu_i(\{0\}) = \mu_i(\{1\}) = \frac{1}{2} \) for \( i = 1, 2, 3 \). However, there is no triple \( (X_1, X_2, X_3) \) such that \( \mu_{ij} \) is the joint distribution of \( (X_i, X_j) \) since with probability 1, \( X_1, X_2 \) and \( X_3 \) are pairwise different. This contradicts with the fact that only 2 values (0 and 1) are accessible for three random variables.

6. We use extensively the “conditioning” interpretation of conditional independence. Let \( f \) be a bounded measurable function. \( X \) and \( Y \) are conditionally independent given \( Z \) means that \( \mathbb{E}(f(X)|Y, Z) \overset{(\ast)}{=} \mathbb{E}(f(X)|Z) \). \( X \) and \( Z \) are conditionally independent given \( \mathcal{F} \) suggests that \( \mathbb{E}(f(X)|Z) \overset{\ast\ast}{=} \mathbb{E}(f(X)|\mathcal{F}) \) since \( \mathcal{F} \subset \sigma(Z) \). By tower property of conditional expectation,

\[
\mathbb{E}(f(X)|Y, \mathcal{F}) = \mathbb{E}[\mathbb{E}(f(X)|Y, Z)|Y, \mathcal{F}] \overset{\#}{=} \mathbb{E}[\mathbb{E}(f(X)|\mathcal{F})|Y, \mathcal{F}] = \mathbb{E}(f(X)|\mathcal{F}),
\]

where \( (\#) \) follows \( (*) \) and \( (\ast\ast) \). Therefore, \( X \) and \( Y \) are conditionally independent given \( \mathcal{F} \).