205B homework, #4; due Tuesday February 20

Durrett Chapter 6 Exercises 6.6, 6.7

5. Let (X_n) be an irreducible Markov chain on S with transition matrix $(p(x,y))$. Let B be a finite subset of S such that the chain a.s. visits B infinitely often. Let (Z_m) be the chain watched only on B:

$$Z_m = X_{T_m}; \quad T_m = \min\{n > T_{m-1} : X_n \in B\}.$$

Then Z is irreducible, and so has stationary distribution $\hat{\pi}$, say. Define

$$\mu(x,y) = \mathbb{E}_x \sum_{n=0}^{\infty} 1_{(X_n = y, T_B > n)}, \quad x \in B,\ y \in S.$$

$$\pi(y) = \sum_{x \in B} \hat{\pi}(x)\mu(x,y).$$

Show that π is invariant, in the sense

$$\pi(y) = \sum_{z \in S} \pi(z)p(z,y) \leq \infty,\ y \in S.$$

6. A population consists of X_n individuals at times $n = 0, 1, 2, \ldots$. Between time n and time $n+1$ each of these individuals dies with probability p independently of the others; and the population at time $n+1$ consists of the survivors together with an independent random (Poisson (λ)) number of immigrants.

Let X_0 have arbitrary distribution. What happens to the distribution of X_n as $n \to \infty$? [Hint: consider first the case where X_0 has Poisson (λ_0) distribution]

7. Let X_n be the Markov chain on states $0, 1, \ldots, K$ with transition matrix

$$p(i, i+1) = 2/3 \text{ and } p(i, i-1) = 1/3;\ 1 \leq i \leq K-1$$

$$p(0,0) = p(K,K) = 1$$

and initial state $i_0 \neq 0, K$. Let X_n^* be the process X_n conditioned on the event $\{X_m = K \text{ ultimately}\}$.
(a) Prove carefully that X_n^* is Markov.
(b) Find its transition matrix.
(c) Find the distribution of $\min_{n \geq 0} X_n^*$.