205B homework, #4; due Tuesday February 20

Durrett Chapter 6 Exercises 6.6, 6.7

5. Let \((X_n)\) be an irreducible Markov chain on \(S\) with transition matrix \((p(x,y))\). Let \(B\) be a finite subset of \(S\) such that the chain a.s. visits \(B\) infinitely often. Let \((Z_m)\) be the chain watched only on \(B\):

\[Z_m = X_{T_m}; \quad T_m = \min\{n > T_{m-1} : X_n \in B\} \]

Then \(Z\) is irreducible, and so has stationary distribution \(\hat{\pi}\), say. Define

\[\mu(x,y) = \mathbb{E}_x \sum_{n=0}^{\infty} 1_{\{X_n = y, T_B > n\}}, \quad x \in B, \ y \in S.\]

\[\pi(y) = \sum_{x \in B} \hat{\pi}(x) \mu(x,y).\]

Show that \(\pi\) is invariant, in the sense

\[\pi(y) = \sum_{z \in S} \pi(z) p(z,y) \leq \infty, \ y \in S.\]

6. A population consists of \(X_n\) individuals at times \(n = 0,1,2,\ldots\). Between time \(n\) and time \(n+1\) each of these individuals dies with probability \(p\) independently of the others; and the population at time \(n+1\) consists of the survivors together with an independent random (Poisson (\(\lambda\))) number of immigrants.

Let \(X_0\) have arbitrary distribution. What happens to the distribution of \(X_n\) as \(n \to \infty\)? [Hint: consider first the case where \(X_0\) has Poisson (\(\lambda_0\)) distribution]

7. Let \(X_n\) be the Markov chain on states \(0,1,\ldots,K\) with transition matrix

\[p(i,i+1) = 2/3 \quad \text{and} \quad p(i,i-1) = 1/3; \ 1 \leq i \leq K-1\]

\[p(0,0) = p(K,K) = 1\]

and initial state \(i_0 \neq 0,K\). Let \(X_n^*\) be the process \(X_n\) conditioned on the event \(\{X_m = K\ \text{ultimately}\}\).

(a) Prove carefully that \(X_n^*\) is Markov.
(b) Find its transition matrix.
(c) Find the distribution of \(\min_{n \geq 0} X_n^*\).