205B Homework #1, due Tuesday 30 January

[Theorem 7 and Corollary 8 refer to the notes linked to the January 18 class.]

1. Suppose probability measures satisfy $\pi \ll \nu \ll \mu$. Show that

$$\frac{d\pi}{d\mu} = \frac{d\pi}{d\nu} \times \frac{d\nu}{d\mu}.$$

2. In the setting of Theorem 7 [hard part], where S_2 is nice, show that Q is unique in the following sense. If Q^* is another conditional probability kernel for μ, then

$$\mu_1\{x : Q^*(x, B) = Q(x, B) \text{ for all } B \in \mathcal{S}_2\} = 1.$$

3. Let F be a distribution function. Let $c > 0$. Find a simple formula for

$$\int_{-\infty}^{\infty} (F(x + c) - F(x))\, dx.$$

4. In the proof of Corollary 8 we used the inverse distribution function

$$f(x, u) = \inf\{y : u \le Q(x, (-\infty, y])\}$$

associated with the kernel Q. Show that f is product measurable.

5. Given a triple (X_1, X_2, X_3), we can define 3 p.m.'s $\mu_{12}, \mu_{13}, \mu_{23}$ on $\mathbb{R}^2$ by

$$\mu_{ij} \text{ is the distribution of } (X_i, X_j). \tag{1}$$

These p.m.'s satisfy a consistency condition:

$$\begin{gathered}\text{the marginal distribution } \mu_1 \text{ obtained from } \mu_{12} \text{ must coincide} \\ \text{with the marginal obtained from } \mu_{13}, \text{ and similarly for } \mu_2 \text{ and } \mu_3.\end{gathered} \tag{2}$$

Give an example to show that the converse is false. That is, give an example of $\mu_{12}, \mu_{13}, \mu_{23}$ satisfying (2) but for which there does not exist a triple (X_1, X_2, X_3) satisfying (1).

6. Suppose X and Y are conditionally independent given Z. Suppose X and Z are conditionally independent given $\mathcal{F}$, where $\mathcal{F} \subseteq \sigma(Z)$. Prove that X and Y are conditionally independent given $\mathcal{F}$.