1 Introduction

These problems are meant to be practice problems for you to see if you have understood the material reasonably well. They are neither exhaustive (e.g. Diffusions, continuous time branching processes etc are not covered) nor are they meant to cause extra stress. Some sections have more problems than the others but this is not meant to reflect relative importance of the various Chapters in the book, rather it reflects the ease with which questions on particular topics can be found. Try the problems and think about as many problems as you can but don't worry if you don't get time to try some/any of the problems. Try to revise the class material well, as well as the homework problems and make sure you have understood the concepts clearly. You are of course free to email us (aldous@stat, shanky@stat) with questions and we can try to give you hints.

2 Basic notions

1. Let X and Y be independent exponential random variables with rate α. Find the densities of the random variables $X^3, |X-Y|, \min(X, Y^3)$

2. It is assumed that the lifetimes of electric bulbs have an exponential distribution with an unknown expectation α^{-1}. To estimate α a sample of r bulbs is taken and one observes the lifetimes of these bulbs $X_{(1)} < X_{(2)} < \ldots < X_{(r)}$. The *best linear unbiased estimator* of α^{-1} is a linear combination $U = \sum_i \lambda_i X_{(i)}$ such that $E(U) = \alpha^{-1}$ and Var(U) is the smallest possible. Show that:

$$U = \frac{n-r}{r} X_{(r)} + \frac{1}{r} \sum_{i=1}^{r} X_{(i)}$$

   and then $Var(U) = \frac{1}{r}.\alpha^{-2}$

3. In $\mathbb{R}^2$ let the 2 co-ordinate axes be denoted by the x-axis and the y-axis. An isoceles triangle is formed by a unit vector in the (positive) x-direction and another unit vector in a random direction. Find the distribution of the length of the third side.

4. Let F be a cumulative distribution function which is *continuous and strictly increasing*. Let $X \sim U[0, 1]$ and define $Y = F^{-1}(X)$. Compute the distribution function of Y.

5. Let X_i $1 \le i \le 3$ be IID U[0,1] and let the corresponding order statistics be $X_{(i)}$. Let $U_1 = \frac{X_{(1)}}{X_{(2)}}$ and $U_2 = \frac{X_{(2)}}{X_{(3)}}$. Show that U_1, U_2 are independent.

6. Toss a dice 7 times. Let S_7 be the sum of the faces shown on the dice (i.e $S_7 = \sum_1^7 X_i$, X_i being the face shown on the i^{th} toss. Compute $P(S_7 = 10)$.

7. Let X, Y be independent Poisson random variables with mean μ, λ respectively. Find:
   - Distribution of $X + Y$
   - Conditional distribution of X given that $X + Y = n$.

8. Let X be a real valued random variable and let f, g be 2 monotonically increasing bounded functions. Prove the following *non negative correlation fact*

$$E(f(X)g(X)) \geq E(f(X)).E(g(X))$$

9. Let X_n be the size of the population at time n of a Branching process which starts at time 0 with 1 individual and for which the offspring distribution has mean μ. What is $E(X_n.X_m)$ for $n > m$

3 Inequalities

1. Let X_1, X_2 be independent and identically distributed. Show that

$$P(|X_1 - X_2| > t) \leq 2.P(|X_1| > \frac{1}{2}.t)$$

2. Suppose a random variable has moments of all order. Define

$$u(t) = \log E(|X|^t) \; t \geq 0$$

Show that u is a convex function of t.

4 Combinatorics and Combinatorial optimization

1. Suppose that each of n sticks is broken into one long and one short part. The 2n parts are arranged into n pairs from which new sticks are formed. Find the probability that the parts will be joined in the original order.

2. In the setting of the previous problem find the probability that all long pairs are paired with short parts.

3. Two equivalent decks of N different cards each are put into random order and matched against each other. If a card occupies the same place in both decks we say that a matching has occurred. What is the probability that there is at least one match ?

4. Suppose there are n players playing a tournament, where each player plays every other player once and every game has an outcome(i.e win or loss ,no draws are allowed). We represent the tournament via a directed graph where there are n nodes numbered $\{1, \ldots n\}$(representing the n players) and an oriented edge between every pair, i.e every edge is of the form of the *ordered pair* (x,y) (think of it as an edge pointing from node x to node y) iff x defets y. Fix k and say that a tournament has property S_k if for any subset of k players, there is some player who beats them all (i.e $\forall$ distinct $i_1, i_2, \ldots, i_k$ there is some $m \in \{1, \ldots, n\}/\{i_1, \ldots i_k\}$ such that player m beats all the i_j). Is it true that for n large enough there is a tournament with property S_k ? Show that if $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$ then there is a tournament on n players with the property S_k

5. Let $X \sim N(0, \sigma_1^2)$ and $Y \sim N(0, \sigma_2^2)$ where $0 < \sigma_1^2 < \sigma_2^2$. Show that $\forall x > 0$

$$P(X^2 > x) < P(Y^2 > x)$$

Note the strict inequality.

5 Poisson processes

1. Let $W_i \sim U[0,1]$. Let N be the number of indices i satisfying $t < \Pi_1^i(W_j) < 1$. Find the distribution of N.

2. Let $X(t)$ be a homogeneous Poisson process with parameter λ. determine the covariance between $X(t)$ and $X(t+\tau)$ where $t > 0, \tau > 0$.

3. Consider 2 independent homogeneous Poisson process $X(t), Y(t)$ with parameters λ and μ respectively. Let 2 succesive points of the $X(t)$ occur at the random times T and T' so that $X(t) = X(T)$, $T \leq t < T'$ and $X(T') = X(T) + 1$. Define N $=$ $Y(T') - Y(T)$ be the number of points of Y in the time interval (T, T'). Find the distribution of N

4. In a certain town at time $t = 0$ there are no bears. Brown and Grizzly bears arive as independent Poisson processes B(.) and G(.) independently with respective intensities β and γ

   - Show that the chance that the first bear is brown is $\beta/(\beta+\gamma)$
   - Find the probability that between 2 consecutive brown bears, there are exactly r grizzly bears.
   - Given that $B(1) = 1$, find the expected value of the time at which the first bear arrived.

5. The r^{th} point T_r of a Poisson process N of constant intensity λ on $\mathbb{R}^+$ gives rise to an effect $X_r e^{-\alpha(t-T_r)}$ at time $t \geq T_r$ where the X_i are i.i.d with finite variance (independent of the point process as well). Find the mean and variance of the total effect $S(t) = \sum_1^{N(t)} X_r e^{-\alpha(t-T_r)}$ in terms of the first two moments of X_r.

6 Markov Chains

1. Let X_n be the amount of water in a reservoir at noon on day n. During the 24 hour period beginning at this time, a quantity Y_n of water flows into the reservoir, and just before noon on each day exactly one unit of water is removed (if this amount can be found). The maximum capacity of the reservoir is K, and excessive inflows are spilled and lost. Assume that the Y_n are independent and identically distributed random variables and that, by rounding off some laughably small unit of volume, all numbers in this exercise are non negative integers. Show that (X_n) is a Markov Chain, and

find its transition matrix and an expression for its stationary distribution in terms of the probability generating function G of Y_n. Find the stationary distribution when Y has probability generating function $G(s) = p(1-qs)^{-1}$.

2. Suppose a virus can exist in N different strains and in each generation either stays the same, or with probability α mutates to another strain which is chosen at random. What is the probability that the strain in the n^{th} generation is the same as that in the 0^{th} ?

3. Which of the following are reversible Markov Chains :

   (a) Chain $X = \{X_n\}$ having transition matrix P:

   $$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

   with $1 > \alpha > 0, 1 > \beta > 0$.

   (b) The chain $X = \{Y_n\}$ having transition matrix P:

   $$P = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$

   where $0 < p < 1$

   (c) $Z_n = (X_n, Y_n)$ where X_n is the chain in 3a and Y_n is the chain in 3b

7 Branching processes

1. Suppose we have a large geographical area, in which for each subarea, a branching process characterized by the probability generating function of a Poisson distribution with parameter λ is taking place. We assume furthermore that the value of λ varies depending on the subarea and its distribution over the whole area is that of a gamma. Formally we postulate that given λ the chance that an individual has k children is given by:

   $$P(Z_1 = k|\lambda) = e^{-\lambda}\lambda^k/k!$$

   where λ itself is a random variable according to a gamma distribution with density function

   $$f(\lambda) = \frac{(q/p)^\alpha \lambda^{\alpha-1}}{\Gamma(\alpha)} \exp\left(-\frac{q}{p}.\lambda\right), \ \lambda \geq 0$$

   Thus the unconditional distribution of Z_1 is

   $$P(Z_1 = k) = \int_0^\infty f(\lambda) P(Z_1 = k|\lambda) d\lambda$$

Prove that the generating function of the above branching process is given by:

$$E(s^{Z_1}) = \left(\frac{q}{1-ps}\right)^{\alpha}$$

which is negative binomial.

2. Suppose that in a branching process with $Z_0 = 1$, the number of offspring of an initial particle has a distribution whose generating function is $f(s)$ with mean μ. Each member of the first generation has a number of offspring whose distribution has generating function $g(s)$ with mean ν. The next generation has generating function f, the next g, and the functions continue to alternate in this way from generation to generation.

   - Define constants c_n in terms of μ, ν so that $c_n Z_n$ is a martingale.
   - What can you say about the eventual behavior of the Branching process when

$$\mu.\nu < 1$$

8 Martingales

1. Let W(n) be a branching process with immigration:

$$W(n+1) = Y_n + \sum_{1}^{W(n)} X_{n,i}$$

   where Y_n is the immigration in generation n and $X_{n,i}$ is the number of offspring of the i^{th} individual in generationn , all independent. Suppose $E[Y_n] = \lambda$ and $E[X_n, j] = m \neq 1$. Show that

$$Z_n = \frac{1}{m^n}[W(n) - \lambda\left(\frac{m^n - 1}{m-1}\right)]$$

   is a martingale.

2. Assume Y_i are i.i.d with $P(Y_i = 1) = p = 1 - P(Y_i = -1)$. Fix positive integers a and b. With $S_0 = 0$ and $S_n = \sum_1^n Y_i$, let

$$T = \inf\{n : S_n = -a \text{ or } S_n = b\}$$

   Establish:

$$E[T] = \frac{b}{p-q} - \frac{a+b}{p-q}.\frac{1-(p/q)^b}{1-(p/q)^{a+b}}$$

   when $p \neq q$

3. Consider a family of r.v.'s X_n each having finite absolute expectation and satisfying

$$E[X_{n+1}|X_0, \dots X_n] = \alpha X_n + \beta X_{n-1},\ n > 0$$

   with $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Find an appropriate value of a such that the sequence

$$Y_n = aX_n + X_{n-1}\ n \geq 1$$

   with $Y_0 = X_0$, constitues a martingale with respect to $\{X_n\}$.

4. Let $\{\xi_i\}$ be a sequence of real valued jointly distributed random variables that satisfy $E[\xi_i|\xi_o, \dots \xi_{i-1}] = 0$ $i = 1, 2, \dots$ Define $X_0 = \xi_0$

$$X_{n+1} = \sum_0^n \xi_{i+1} f_i(\xi_0, \dots \xi_i)$$

where f_i are some prescribed sequence of functions of $i+1$ real variables. Show that X_n form a martingale.

5. Consider a Markov chain $\{X_n\}$ on state space $\{0, 1, \dots N\}$ with transition matrix

$$P_{ij} = \frac{\binom{2i}{j} . \binom{2N-2i}{N-j}}{\binom{2N}{N}}$$

Determine λ so that $\frac{X_n(N-X_n)}{\lambda^n}$ is a martingale.

6. Suppose Y_0 is uniformly distributed on $(0, 1]$ and given Y_n, suppose Y_{n+1} is uniformly distributed on $(1 - Y_n, 1]$. Show X_0 and

$$X_n = 2^n . \Pi_{k=1}^n \left(\frac{1 - Y_k}{Y_{k-1}} \right)$$

is a martingale.

7. Let $P_{ij} = e^{-i} i^j / j!$ $i, j = 0, 1, \dots$ be the transition probabilities for a markov Chain X_n. We define $P_{00} = 1$

   - Verify that X_n is a martingale
   - Derive the inequality

$$P(\max_{0 \leq n \leq \infty} X_n \geq a | X_0 = i) \leq i/a$$

   - prove that $\lim_{n \to \infty} X_n = 0$ with probability one from any starting point.