STATISTICS 150 Homework problems on martingales

1. In a Galton-Watson branching process, starting from one individual, let $Z(n)$ be the population size in generation n and let $0 < q < 1$ be the probability of eventual extinction. Show that

$$M_n = q^{Z(n)}$$

is a martingale.

[Hint: recall the equation that determines q]

2. Suppose (X_n) and (Y_n) are martingales with respect to the same filtration. Are the following processes necessarily martingales?
(a) $Z_n = X_n + Y_n$
(b) $W_n = \max(X_n, Y_n)$

3. Suppose $(X_n, 0 \leq n < \infty)$ are adapted to $(\mathcal{F}_n)$ and satisfy

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = \alpha X_n + \beta X_{n-1},\ n \geq 1$$

for constants $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Find a constant a such that, for $Y_0 = X_0$ and

$$Y_n = aX_n + X_{n-1},\ n \geq 1$$

the process (Y_n) is a martingale.