Lecture 7

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Ideas used in Lecture 6.

Markov chain $(X_t, t = 0, 1, 2 \ldots)$ with transition matrix $\mathbf{P}$.

- $\mu(t) = \mu(0)\mathbf{P}^{(t)}$ gives distribution $\mu(t)$ of X_t.
- $f(t) = \mathbf{P}^{(t)} f$ gives $f_i(t) = \mathbb{E}_i f(X_t)$.
- $h(i) = \mathbb{E}_i T_A$ can be found as solution of first-step equation $h(i) = 1 + \sum_j p_{ij} h(j)$.
- $g(i) = \mathbb{P}_i(T_A < T_B)$ can be found as solution of first-step equation $g(i) = \sum_j p_{ij} g(j)$.

There are some special chains where one **can** find analytic solutions to these first-step equations. I will do some in this lecture, and others are in the homework.

Example: Simple asymmetric random walk – "Gambler's Ruin".
In words, you start with i dollars and bet 1 dollar each step, with probability p of winning, and continue until reach K or 0.

States $\{0, 1, \ldots, K\}$

$$p_{i,i+1} = p, \;\; p_{i,i-1} = 1 - p, \;\; 1 \leq i \leq K - 1$$

and states 0 and K are absorbing:

$$p_{00} = 1, \;\; p_{KK} = 1.$$

We study

$$g(i) = \mathbb{P}_i(T_K < T_0); \quad h(i) = \mathbb{E}_i T_{\{0,K\}}.$$

In the symmetric case $p = 1/2$ we already know

$$g(i) = i/K, \quad h(i) = i(K - i).$$

In the asymmetric case $p \neq 1/2$ we can find the formulas

$$g(i) = \mathbb{P}_i(T_K < T_0) = \frac{(\frac{1-p}{p})^i - 1}{(\frac{1-p}{p})^K - 1}, \ 0 \leq i \leq K.$$

$$h(i) = \frac{i}{1-2p} - \frac{K}{1-2p} \times \frac{(\frac{1-p}{p})^i - 1}{(\frac{1-p}{p})^K - 1}, \ 0 \leq i \leq K.$$

I will outline the argument on the board "knowing general form of solution to look for". See [PK] section 3.6 for full proof, in slightly different symbols.

Numerical example. Suppose you attempt to double your money by betting $1 on red at roulette, where p(win) = 18/38. The table shows (i = initial fortune, K = target) probability of success and mean number of plays, comparing p(win) = 18/38 with the "fair" p(win) = 1/2.

	Probability double your money		$\mathbb{E}$(number of plays)	
	$p = \frac{1}{2}$	$p = \frac{18}{38}$	$p = \frac{1}{2}$	$p = \frac{18}{38}$
i = 10, K = 20	50%	26%	100	92
i = 20, K = 40	50%	11%	400	298
i = 100, K = 200	50%	$\frac{1}{40{,}000}$	10,000	1,900

Example: success runs.

Here the states are $\{0, 1, 2, \ldots\}$ and the transition probabilities are of the form

$$p_{i,i+1} = q_i, \ \ p_{i,0} = 1 - q_i$$

where $0 < q_i < 1$.

We can use the connectivity structure (of the graph of possible transitions) in this example to calculate $H(k) = \mathbb{E}_0 T_k$. We find

$$H(k) = \frac{1}{q_0 q_1 q_2 \ldots q_{k-1}} + \frac{1}{q_1 q_2 \ldots q_{k-1}} + \ldots + \frac{1}{q_{k-1}}.$$

[work on board: also [PK] section 3.5]

Example: Death and Immigration process.

Imagine X_t = population size at time t. Between times t and $t+1$, each individual may die independently with probability p, and a random Poisson(λ) distributed number of immigrants arrive.

So states are $\{0, 1, 2, \ldots\}$ and by considering the number of survivors k

$$p_{ij} = \sum_{k=0}^{\min(i,j)} \binom{i}{k} (1-p)^k p^{i-k} \times \frac{e^{-\lambda} \lambda^{j-k}}{(j-k)!}.$$

Suppose the initial distribution of X_0 is Poisson(λ_0). From a STAT134 fact, the number of survivors to time 1 has Poisson($\lambda_0 q$) distribution, for $q = 1-p$. So X_1 also has Poisson distribution with mean $\lambda_1 = \lambda_0 q + \lambda$. Inductively X_t also has Poisson distribution with mean

$$\lambda_t = q\lambda_{t-1} + \lambda$$

from which we can calculate

$$\lambda_t = \lambda_0 q^t + \lambda \sum_{s=0}^{t-1} q^s.$$

As $t \to \infty$ we have $\lambda_t \to \lambda \sum_{s=0}^{\infty} q^s = \lambda/p$.

So without needing complicated calculations, in this example the distribution of X_t converges as $t \to \infty$ to a limit distribution, which is the Poisson(λ/p) distribution.