Lecture 35

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The M/G/1 queue model is

- Customers arrive at times of a rate-λ Poisson point process
- Service times $Y_1, Y_2, \ldots$ are IID.
- Write $\nu = \mathbb{E}Y$ and note that $1/\nu$ is "service rate".
- $X(t) =$ number of customers at time t.
- 1 server.

Here $(X(t), 0 \leq t < \infty)$ is **not** a continuous-time Markov chain. But we can do some calculations.

Idle and busy periods. The server alternates idle and busy periods – lengths $I_1, B_1, I_2, B_2, \ldots$. We know $\mathbb{E}I_1 = 1/\lambda$. How do we calculate $\mathbb{E}B_1$?

Method 1. Recall the **cycle trick**. Write $T_1, T_2, \ldots$ for the times between successive "renewals", suppose there is a "reward" R_i associated with the renewal interval T_i, and suppose the sequence of pairs $(T_i, R_i), i = 1, 2, \ldots$ is IID. Then

$$\text{long-run average reward per unit time} = \mathbb{E}R/\mathbb{E}T.$$

To use this, write $T_i = I_i + B_i$ and $R_i = B_i$. So long-run average proportion of time server is busy $= \mathbb{E}B_1/(\mathbb{E}I_1 + \mathbb{E}B_1)$.

Demand for service per unit time $= \lambda \times \mathbb{E}Y =$ proportion of time server is busy. So

$$\lambda\nu = \frac{\mathbb{E}B_1}{1/\lambda + \mathbb{E}B_1}$$

We solve to get

$$\mathbb{E}B_1 = \frac{\nu}{1 - \lambda\nu}.$$

Method 2. Consider
$N^* =$ number of arrivals during a service period
and argue [board]

$$\mathbb{E}(B_1 | N^* = n) = \nu + n\mathbb{E}B_1.$$

Then [board]

$$\mathbb{E}B_1 = \nu + (\mathbb{E}N^*)\ \mathbb{E}B_1$$

which we can solve to get

$$\mathbb{E}B_1 = \frac{\nu}{1 - \mathbb{E}N^*}.$$

Then

$$\mathbb{E}(N^* | Y_1 = t) = \lambda t$$

and so

$$\mathbb{E}N^* = \lambda\nu.$$

Here is a third method which calculates something different. There is a **discrete-time** Markov chain associated with the M/G/1 queue:
$X_n =$ number of customers just after the departure of the n'th customer.

Here [board]

$$X_{n+1} = \max(X_n - 1, 0) + A_n$$

$$A_n = \text{ number of arrivals during next service period}$$

and the A_n are IID.

As a special property of the M/G/1 queue, the stationary distribution of (X_n) is the same as the equilibrium distribution of "number of customers" in the original queue process. Using this fact we can calculate the expectation of "number of customers".

[calculation on board – follows [PK] sec. 9.3.1].