Lecture 33

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Martingales associated with Brownian motion

For standard Brownian motion $(B(t), 0 \leq t < \infty)$ the following processes are continuous-time martingales.

- $B(t)$
- $B^2(t) - t$
- $B^3(t) - 3tB(t)$
- $B^4(t) - 6tB^2(t) + 3t^2$
- ......
- $\exp(\theta B(t) - \theta^2 t/2), \quad$ for fixed $-\infty < \theta < \infty$

By using the optional sampling theorem – for a martingale $M(t)$ and a stopping time T, under mild extra conditions

$$\mathbb{E}M(T) = \mathbb{E}M(0)$$

we can derive formulas for BM.

We can repeat results for simple symmetric random walk.
Take $-a < 0 < b$ and consider

$$T_{a,b} = \min\{t : B(t) = -a \text{ or } b\}.$$

- $\mathbb{P}(B(T_{a,b}) = b) = \frac{a}{a+b}, \quad \mathbb{P}(B(T_{a,b}) = -a) = \frac{b}{a+b}$
- $\mathbb{E} T_{a,b} = ab$

Note the first result can be rewritten as

$$\mathbb{P}(T_b < T_{-a}) = \frac{a}{a+b} \tag{1}$$

and this is true for any continuous-path martingale with $M(0) = 0$.

Fix $\mu, \sigma > 0$ and consider BM with drift rate $-\mu$ and variance rate σ^2:

$$X(t) = \sigma B(t) - \mu t.$$

Because $X(t) \to -\infty$ as $t \to \infty$ there is some maximum value

$$S = \sup_{0 \le t < \infty} X(t) < \infty.$$

We can find the distribution of S by considering the value of θ for which

$$M(t) = \exp(\theta X(t))$$

is a martingale. This works out [board] as

$$\theta = 2\mu/\sigma^2.$$

A hitting time

$$T_b^* = \inf\{t : X(t) = b\}$$

for $X(t)$ is the same as the hitting time

$$T_{e^{\theta b}} = \inf\{t : M(t) = e^{\theta b}\}$$

for $M(t)$. Apply (1) to $M(t) - 1$: [board]

Fix $\mu, \sigma > 0$ and consider BM with drift rate $-\mu$ and variance rate σ^2:

$$X(t) = \sigma B(t) - \mu t.$$

$$T_b^* = \inf\{t : X(t) = b\}$$

Set $\theta = 2\mu/\sigma^2$,

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For $-a < 0 < b$,

$$\mathbb{P}(T_b^* < T_{-a}^*) = \frac{1 - \exp(-\theta a)}{\exp(\theta b) - \exp(-\theta a)}.$$

Letting $a \to \infty$ we have, for

$$S = \sup_{0 \le t < \infty} X(t) < \infty$$

that

$$\mathbb{P}(S \ge b) = \mathbb{P}(T_b^* < \infty) = \exp(-\theta b), 0 < b < \infty.$$