Lecture 32

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16 November 2015

A family of RVs $(G(t))$ is called **Gaussian** or a **Gaussian process** if the joint distribution of any finite set of these RVs is multivariate Normal. From STAT134 facts about multivariate Normal distributions we can read off facts about Gaussian processes.

- If a process is known to be Gaussian, then its distribution is determined by its mean/covariance structure, that is by the functions $\mathbb{E}G(t)$ and $cov(G(s), G(t))$.
- A RV defined as a linear combination $X = \sum_i a_i G(t_i)$ or $X = \int a(t)G(t)\, dt$ has Normal distribution, and including X in the original family $(G(t))$ preserves the Gaussian property.
- Within a Gaussian family, if one RV is uncorrelated with a subfamily, then it is independent of that subfamily.

Standard BM $(B(t), 0 \leq t < \infty)$ is the Gaussian process with [board]

$$\mathbb{E}B(t) = 0; \quad \mathbb{E}[B(s)B(t)] = \min(s, t).$$

The **Brownian bridge** process $(B^o(t), 0 \leq t \leq 1)$ is defined to have the (*) conditional distribution of standard BM over $[0, 1]$ given $B(1) = 0$. We can construct (mathematically) this process by a trick. Define

$$(**) \quad B^o(t) = B(t) - tB(1), \quad 0 \leq t \leq 1.$$

Working with this definition we see [board]

- $(B^o(t))$ is a Gaussian process; $B^o(0) = B^o(1) = 0$.
- $\mathbb{E}B^o(t) = 0$.
- $\mathbb{E}[B^o(s)B^o(t)] = s(1-t), \ 0 \leq s \leq t \leq 1$.
- $\mathbb{E}[B^o(t)B(1)] = 0$.

The final point implies that $(B^o(t), 0 \leq t \leq 1)$ is independent of $B(1)$. So the unconditional distribution of $(B^o(t), 0 \leq t \leq 1)$ is the same as its conditional distribution given $B(1) = 0$, and then construction (**) fits the original description (*).

Recall previous results: joint density of $(M(t), B(t))$ is

$$f_{M(t),B(t)}(a,b) = \frac{2(2a-b)}{\sqrt{2\pi}} \ t^{-3/2} \ \exp(-(2a-b)^2/(2t)); \quad a \geq 0, \ a \geq b.$$

This had two interesting consequences.

Proposition

$$\mathbb{P}(M_1 > a | B(1) = 0) = \exp(-2a^2), \ a > 0.$$
$$\mathbb{P}(B(1) \leq -b | M(1) = 0) = \exp(-b^2/2), \ b > 0.$$

The first identity here tell us that for Brownian bridge

$$M^o := \max_{0 \leq t \leq 1} B^o(t)$$

has distribution

$$\mathbb{P}(M^o > a) = \exp(-2a^2), \ a > 0.$$

Brownian bridge arises in Statistics as the scaling limit of empirical distributions.

[board and [PK] sec 8.3.3]

Example. What is the distribution of $V = \int_0^1 a(t)B^o(t)dt$?

We know V has Normal, mean 0, distribution: what is the variance? The "trick" is to write V^2 as

$$V^2 = \left(\int_0^1 a(s)B^o(s)ds \right) \ \left(\int_0^1 a(t)B^o(t)dt \right)$$

so then

$$\mathbb{E}V^2 = \int_0^1 \int_0^1 a(s)a(t) \ s(1-t) \ \ dsdt.$$

Note: this is not "stochastic integration" (stochastic calculus) but is just ordinary calculus.

Somewhat analogous to Brownian bridge, we define **Brownian meander** $(B^+(t), 0 \le t \le 1)$ be the BM $(B(t), 0 \le t \le 1)$ conditioned on $(B(t) \ge 0,\ 0 \le t \le 1)$.

This is harder to study explicitly, but the second formula in the Proposition tells us

$$\mathbb{P}(B^+(1) > b) = \exp(-b^2/2),\ \ b > 0.$$