Lecture 31

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$(B(t), 0 \le t < \infty)$ is standard Brownian motion (BM),

Consider $M(t) = \max_{0 \le s \le t} B(s)$. The joint density of $(M(t), B(t))$ is

$$f_{M(t),B(t)}(a,b) = \frac{2(2a-b)}{\sqrt{2\pi}}\ t^{-3/2}\ \exp(-(2a-b)^2/(2t)); \quad a \ge 0,\ a \ge b.$$

This formula is complicated, but there are two interesting consequences.

Proposition

$$\mathbb{P}(M_1 > a | B(1) = 0) = \exp(-2a^2),\ a > 0.$$
$$\mathbb{P}(B(1) \le -b | M(1) = 0) = \exp(-b^2/2),\ b > 0.$$

[board]

Other calculations we can do involve

$$L = \sup\{t \leq 1 : \ B(t) = 0\}; \quad R = \inf\{t \geq 1 : \ B(t) = 0\}.$$

(Jargon: the **excursion** containing time 1 happens over the interval $[L, R]$.) The results are

$$f_R(t) = \frac{1}{\pi(t-1)^{1/2}t}, \ 1 < t < \infty$$

$$\mathbb{P}(L \leq s) = 2\pi^{-1} \arcsin s^{1/2}, \ 0 < s < 1. \qquad (1)$$

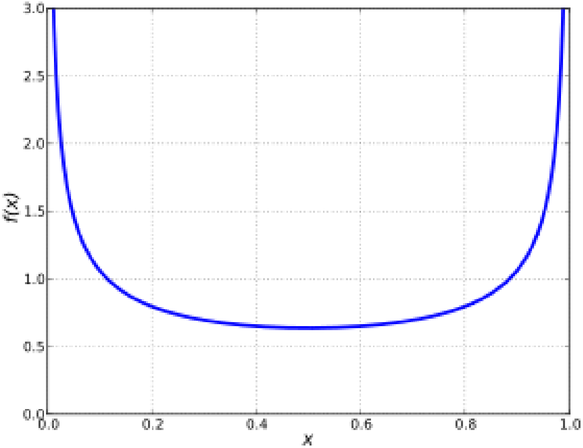
$$f_L(s) = \frac{1}{\pi s^{1/2}(1-s)^{1/2}}, \ 0 < s < 1$$ (**arcsine** distribution).

On the board I will show

$$\mathbb{P}(L \leq s) = \int_{-\infty}^{\infty} g_s(x)\phi_s(x) \ dx$$

$g_s(x) = \mathbb{P}(T_{|x|} > 1 - s) = 1 - 2\bar{\Phi}(|x|/\sqrt{1-s})$, $\phi_s(\cdot)$ is density of $B(s)$.

Then (1) is a hard calculus exercise!


3.0
2.5
2.0
1.5
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f(x)
0.0
0.2
0.4
0.6
0.8
1.0
x