Lecture 30

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It turns out there is a mathematical object $(B(t), 0 \leq t < \infty)$ called "standard Brownian motion" (BM) with properties

1. $B(t)$ has Normal(0,t) distribution.

2. $B(t) - B(s)$ has Normal(0,t-s) distribution $(s < t)$.

3. For $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_k < t_k$ the increments $B(t_1) - B(s_1), \ldots, B(t_k) - B(s_k)$ are independent.

4. The sample paths $t \to B(t)$ are (random) continuous functions of t.

Keep in mind this is a **model** – looking at some time-varying real-world quantity, it may or may not behave like this BM model.

BM is important because

1. one can do many explicit calculations.
2. it is a "building block" for defining other random processes.

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We will study the first hitting time to position $b > 0$

$$T_b = \inf\{t : B(t) = b\}$$

and the maximum up to time t

$$M(t) = \sup\{B(s) : \ 0 \le s \le t\}.$$

Note – an idea we have seen before –

the event $\{M(t) \ge b\}$ is the event $\{T_b \le t\}$.

So the distribution of either T_b or $M(t)$ determines the other:

$$\mathbb{P}(M(t) \ge b) = \mathbb{P}(T_b \le t).$$

We can calculate these distributions – and more – using the **reflection principle**. I will give a more general formulation than [PK] sec. 8.2.1.

Theorem (from general reflection principle)

$$\mathbb{P}(T_b \le t,\ B(t) \ge b+a) = \mathbb{P}(T_b \le t,\ B(t) \le b-a); \quad a, b > 0.$$

[picture on board]

We can deduce quite a lot of information from this identity. Set $a = 0$; we can then argue [board]

$$(*) \qquad \mathbb{P}(M(t) \ge b) = 2\mathbb{P}(B(t) \ge b)$$

which can be rewritten as

$$M(t) =_d |B(t)|.$$

So from last class

$$\mathbb{E}M(t) = \mathbb{E}|B(t)| = \sqrt{2t/\pi}; \quad \mathbb{E}M^2(t) = \mathbb{E}B^2(t) = t.$$

We will find an explicit probability density for T_b below. First note an interesting "paradox". From the Markov property

$$T_{b_1+b_2} =_d T_{b_1} + T_{b_2} \quad \text{(independent)}.$$

But from scaling

$$T_b =_d b^2 T_1.$$

So for integer $k \geq 2$ we have

$$\mathbb{E} T_k = k \times \mathbb{E} T_1; \quad \mathbb{E} T_k = k^2 \times \mathbb{E} T_1.$$

How can this happen?

Use (*) to see

$$\mathbb{P}(T_b \leq t) = \mathbb{P}(M(t) \geq b) = 2\mathbb{P}(B(t) \geq b) = 2\bar{\Phi}(b/t^{1/2}).$$

Differentiate w.r.t. t to get

$$f_{T_b}(t) = b(2\pi)^{-1/2} t^{-3/2} \exp(-b^2/(2t)), \ 0 < t < \infty.$$

[sketch on board] and note $\mathbb{E} T_b = \infty$.

Another calculation will give us the joint density of $(M(t), B(t))$. The reflection principle tells us (changing notation)

$$\mathbb{P}(B(t) \geq a + c) = \mathbb{P}(M(t) \geq a, B(t) \leq a - c); \quad a, c > 0.$$

Set $b = a - c$:

$$\mathbb{P}(M(t) \geq a, B(t) \leq b) = \mathbb{P}(B(t) \geq 2a - b); \quad a > 0, \ a \geq b.$$

Write the right side in terms of $\bar{\Phi}$ and differentiate twice

[calculus on board]
Joint density of $(M(t), B(t))$ is

$$f_{M(t),B(t)}(a,b) = \frac{2(2a-b)}{\sqrt{2\pi}}\ t^{-3/2}\ \exp(-(2a-b)^2/(2t)); \quad a \geq 0,\ a \geq b.$$

This is complicated, but there are two interesting consequences.

Proposition

$$\mathbb{P}(M_1 > a | B(1) = 0) = \exp(-2a^2),\ \ a > 0.$$
$$\mathbb{P}(B(1) \leq -b | M(1) = 0) = \exp(-b^2/2),\ \ b > 0.$$

[board]