Lecture 28

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Other examples of martingales.

1. Consider the rate-λ Poisson counting process $(N(t), 0 \leq t < \infty)$. Here

$$M_t = N(t) - \lambda t$$

is a (continuous-time) martingale. [board]

2. The Polya urn process. Consider a box, initially with $r_0 \geq 1$ red balls and $b_0 \geq 1$ black balls. At each step, pull out a uniform random ball, and the return it into the box along with another new ball of the same color. Consider

$$M_t = \text{ proportion of balls that are red at time } t.$$

Then (M_t) is a martingale. [board]

3. Fisher-Wright genetic model. (2-type, no mutation or selection) (from Lecture 5)

- $2N$ genes in each generation, of types **a** or **A**.
- "children choose parents": each gene is a copy (same type) of a uniform random gene from previous generation.

Then

$$X_t = \text{number of type-}\mathbf{a} \text{ in generation } t$$

is a Markov chain, with states $\{0, 1, 2, \ldots, 2N\}$ and transition probabilities

$$p_{ij} = \mathbb{P}(\mathrm{Bin}(2N, \tfrac{i}{2N}) = j) = \binom{2N}{j} (\tfrac{i}{2N})^j \ \tfrac{2N-i}{2N})^{2N-j}.$$

- -

(X_t) is a martingale.

Here is a counter-intuitive problem.

- regular deck of cards – 26 red and 26 black.
- I deal, face-up.
- At some time you have to bet that the next card will be red.
- If you bet on the first card then $\mathbb{P}(\text{next card is red}) = 1/2$.
- Is there a better strategy (for instance by counting the number of red/black cards dealt)?

Theorem

Whatever strategy you use, $\mathbb{P}(\textit{next card is red}) = 1/2$.

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- A_t event "t'th card is red.
- $\mathcal{F}_t$ = information from first t cards.
- given $\mathcal{F}_t$ the remaining cards are in random order, so $\mathbb{P}(A_{t+1}|\mathcal{F}_t) = \mathbb{P}(A_{52}|\mathcal{F}_t)$.

key idea: betting on next card is like betting on bottom card.

- $M_t = \mathbb{P}(A_{52}|\mathcal{F}_t)$ is a martingale.
- the time τ when we make the bet (on card $\tau + 1$) is a stopping time.
- $\mathbb{P}(\text{win bet}|\mathcal{F}_t, \tau = t) = \mathbb{P}(A_{t+1}|\mathcal{F}_t, \tau = t) = \mathbb{P}(A_{52}|\mathcal{F}_t, \tau = t)$.
- $\mathbb{P}(\text{win bet}|\mathcal{F}_\tau) = M_\tau$
- $\mathbb{P}(\text{win bet}) = \mathbb{E}\mathbb{P}(\text{win bet}|\mathcal{F}_\tau) = \mathbb{E}M_\tau$
- but optional sampling theorem says $\mathbb{E}M_\tau = \mathbb{E}M_0 = \mathbb{E}M_{52} = \mathbb{P}(A_{52}) = 1/2$.

In more advanced probability, we use the optional sampling theorem to prove a variety of **general inequalities** and **general convergence theorems**. I will give one example of each.

The basic property of a martingale was

$$\mathbb{E}(X_{t+1}|\mathcal{F}_t) = X_t.$$

Replacing the equality by an inequality gives two new definitions:

$$\mathbb{E}(X_{t+1}|\mathcal{F}_t) \le X_t. \quad \text{(supermartingale)}$$

$$\mathbb{E}(X_{t+1}|\mathcal{F}_t) \ge X_t. \quad \text{(submartingale)}$$

Theorem

If (X_t) is a supermartingale and $X_t \ge 0$ then

$$\mathbb{P}(\max_{t \le t_0} X_t \ge b) \le \frac{\mathbb{E}X_0}{b}, \ \ b > 0.$$

Letting $t_0 \to \infty$ we can conclude

$$\mathbb{P}(\sup_t X_t \ge b) \le \frac{\mathbb{E}X_0}{b}, \ \ b > 0.$$

Theorem

If (X_t) is a supermartingale and $X_t \geq 0$ then

$$\mathbb{P}(\max_{t \leq t_0} X_t \geq b) \leq \frac{\mathbb{E}X_0}{b}, \ \ b > 0.$$

The event $\{\max_{t \leq t_0} X_t \geq b\}$ is the event $\{\tau \leq t_0\}$ for the stopping time

$$\tau = \min\{t : X_t \geq b\}.$$

The optional sampling theorem for supermartingales says

$$\mathbb{E}X_{\tau^*} \leq \mathbb{E}X_0$$

for stopping times τ^*. Apply to $\tau^* = \min(\tau, t_0 + 1)$.

[continue on board]

Convergence theorems

Simple symmetric random walk $S_n = \sum_{i=1}^n \xi_i$ for IID $\mathbb{P}(\xi = \pm 1) = 1/2$ is a martingale, but clearly there is no finite limit $S_n \to S_\infty$. There are several theorems that say, roughly, that for a "martingale-like" process (X_n),

$$\text{if } \sup_n \mathbb{E}|X_n| < \infty \text{ then } X_n \to \text{ some } X_\infty \text{ a.s.}.$$

Theorem

If (X_n) is a supermartingale and $X_n \geq 0$ then $X_n \to$ some X_∞ a.s.

Theorem

If (X_n) is a submartingale and $\sup_n \mathbb{E} \max(X_n) < \infty$ then $X_n \to$ some X_∞ a.s.

[Note the second theorem implies the first].