# Lecture 25

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- The collection of all events determined by a family $(W, X, Y, Z)$ is called $\sigma(W, X, Y, Z)$
- A RV $T$ whose value is determined by the values of $(W, X, Y, Z)$ is called $\sigma(W, X, Y, Z)$-measurable.
- A $\sigma$-field $\mathcal{F}$ of events is regarded as "information".
- $\mathcal{F} \subseteq \mathcal{G}$ means: if $A \in \mathcal{F}$ then $A \in \mathcal{G}$. So $\mathcal{G}$ contains more information than $\mathcal{F}$.
- A sequence $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ is called a **filtration**. $\mathcal{F}_t$ is the $\sigma$-field of known events at time $t$, and saying a RV $Y$ is $\mathcal{F}_t$-measurable means we know the value of $Y$ at time $t$.

[BZ] exercise 3.1 - [board]

A sequence of coin tosses. $\mathcal{F}_t$ tells us the results of the first $t$ tosses. Which is the smallest $t$ such that $\mathcal{F}_t$ contains the following event.

$C = \{$ the first 100 tosses produce the same outcome $\}$.
$A = \{$ the first occurrence of heads is preceded by at most 10 tails $\}$.
$B = \{$ there is at least one head in the infinite sequence $\}$
$D = \{$ no more than 2 heads and 2 tails in the first 5 tosses $\}$.

Note this is just "logic" – no probability.

For a real-valued RV $X$ and a $\sigma$-field $\mathcal{F}$, we can define the conditional expectation $\mathbb{E}(X|\mathcal{F})$.

- The **gambling interpretation** of $\mathbb{E}(X|\mathcal{F})$ is as the fair stake $Z$ to pay today in order to receive $X$ tomorrow, when $\mathcal{F}$ is the known information,
- The **abstract math definition** of $\mathbb{E}(X|\mathcal{F})$ is as the $\mathcal{F}$-measurable RV $Z$ such that

$$\mathbb{E}[Z 1_A] = \mathbb{E}[X 1_A] \text{ for all } A \text{ in } \mathcal{F}.$$

- In the case $\mathcal{F} = \sigma(Y)$ we have $\mathbb{E}(X|\mathcal{F}) = \mathbb{E}(X|Y)$ as defined before.

Analogous to rules in algebra/calculus, there are many rules for manipulating conditional expectations; will develop as we go.

Most material in this lecture is in [BZ] chapter 3, different notation.

A martingale is a process $(X_0, X_1, X_2, \ldots)$ such that

- $\mathbb{E}(X_{t+1}|\mathcal{F}_t) = X_t$ for each $t \geq 0$.

If no filtration is specified then we take the natural filtration $\mathcal{F}_t = \sigma(X_0, \ldots, X_t)$.

**Examples of martingales** - check on board

**(A).** If $X_1, X_2, \ldots$ are independent, $\mathbb{E}X_i = 0$ and $S_n = \sum_{i=1}^n X_i$ then

$$(0 = S_0, S_1, S_2, \ldots) \text{ is a martingale.}$$

Also, writing $\mathrm{var}\,(X_i) = \sigma_i^2$ and $V_n = S_n^2 - \sum_{i=1}^n \sigma_i^2$.

$$(0 = V_0, V_1, V_2, \ldots) \text{ is a martingale.}$$

**(B).** If $Y_1, Y_2, \ldots$ are independent, $\mathbb{E}Y_i = 1$ and $M_n = \prod_{i=1}^n Y_i$ then

$$(1 = M_0, M_1, M_2, \ldots) \text{ is a martingale.}$$

**(C).** In the Galton-Watson branching process with offspring distribution $\xi$, let $Z_n$ be the population in generation $n$. Write $\mu = \mathbb{E}\xi$. Then

$$(Z_n/\mu^n, \ 0 \le n < \infty) \text{ is a martingale.}$$

**(D).** Given a filtration $(\mathcal{F}_t)$, for **any** RV $X$ with $\mathbb{E}|X| < \infty$ we can consider $M_t = \mathbb{E}(X|\mathcal{F}_t)$ and then

$$(M_t, \ 0 \le t < \infty) \text{ is a martingale.}$$

Recalling $\mathbb{P}(A) = \mathbb{E}1_A$, we can define conditional probability given a $\sigma$-field by $\mathbb{P}(A|\mathcal{F}) = \mathbb{E}(1_A|\mathcal{F})$, and then for **any** event $A$

$$(\mathbb{P}(A|\mathcal{F}_t), \ 0 \le t < \infty) \text{ is a martingale.}$$

Later we'll see how this works with real-world future events, Also relevant to mathematical study of models. Recall (Lecture 6) "first step analysis" of a Markov chain $(X_t)$ with $\mathbf{P} = (p_{ij})$.

Consider disjoint subsets $A, B$ of States – maybe $A = \{a\}$ and $B = \{b\}$. Let's study

$$g(i) = \mathbb{P}_i(T_A < T_B)$$

the probability starting at $i$ of hitting $A$ before hitting $B$. We have

$$g(i) = 1, \ i \in A; \quad g(i) = 0, \ i \in B \tag{1}$$

and by conditioning on the first step

$$g(i) = \sum_j p_{ij} g(j), \ i \notin A \cup B. \tag{2}$$

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In Lecture 6 we discussed solving these equations. Here we observe (explain $T_{A \cup B}$ later)

**(E).**

$$(g(X_t), \ 0 \leq t \leq T_{A \cup B}) \text{ is a martingale.}$$

Conceptual point: "solving these equations" is the same as "find a function $g$ : States $\to \mathbb{R}$ such that $g(X_t)$ is a martingale".

## Rules for manipulating conditional expectation

All RVs assumed integrable.

**1.** $\mathbb{E}(X \pm Y|\mathcal{F}) = \mathbb{E}(X|\mathcal{F}) \pm \mathbb{E}(Y|\mathcal{F})$

**2.** If $X$ is $\mathcal{F}$-measurable then $\mathbb{E}(X|\mathcal{F}) = X$.

**3.** If $X$ is independent of $\mathcal{F}$ then $\mathbb{E}(X|\mathcal{F}) = \mathbb{E}X$.

**4.** If $W$ is $\mathcal{F}$-measurable then $\mathbb{E}(WX|\mathcal{F}) = W\mathbb{E}(X|\mathcal{F})$.

**5.** If $\mathcal{F} \subseteq \mathcal{G}$ then $\mathbb{E}[\mathbb{E}(X|\mathcal{G})\ |\mathcal{F}] = \mathbb{E}(X|\mathcal{F})$. In particular $\mathbb{E}[\mathbb{E}(X|\mathcal{G})] = \mathbb{E}X$.