Lecture 24

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This lecture: outline some "abstract" ideas – details in [BZ] Chapters 1, 2 – as background to the Martingale section.

Suppose we are given a family of RVs $\{W, X, Y, Z\}$. An event such as

$$A = \{W + X^2 < Z - 7\}$$

is "determined" by the family – if we know the values of the family then we know whether A happens.

Definition: The collection of all events determined by the family is called
(in symbols) $\sigma(W, X, Y, Z)$
(in words) "the σ-field generated by the family $\{W, X, Y, Z\}$".

Now suppose we are given a family $\{W, X, Y, Z\}$ and another RV T.
The two assertions below are equivalent:

- $T = g(W, X, Y, Z)$ for some function g
- The value of T is determined by the values of (W, X, Y, Z)

We say "T is $\sigma(W, X, Y, Z)$-measurable".

Different areas within the Mathematical Sciences use the word "information" with different meanings. We will use the meaning

Information is a σ-field of events.

Intuitively, at some past time there were a lot of events A which were uncertain, that is $0 < \mathbb{P}(A) < 1$. But now (time t) some of these events are "known" – we know A happened or did not happen. The collection of all known events is the σ-field which represents the "information" we have at time t.

We use symbols like $\mathcal{F}$ or $\mathcal{G}$ to denote σ-fields. Note

$$\mathcal{F} \subseteq \mathcal{G} \text{ means: if } A \in \mathcal{F} \text{ then } A \in \mathcal{G}$$

That is $\mathcal{F}$ is a smaller collection than $\mathcal{G}$.

For theory, we often write $\mathcal{F}_t$ for the σ-field of known events at time t, without specifying it explicitly. This theory assumes we never forget, so

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$$

and such a sequence is called a **filtration**. In this context, saying a RV Y is $\mathcal{F}_t$-measurable means we know the value of Y at time t.

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In specific examples we usually start with random variables and use them to define the filtration. In particular given RVs $X_0, X_1, X_2, \ldots$ we can define

$$\mathcal{F}_t = \sigma(X_0, \ldots X_t) \quad \text{the } \textbf{natural} \text{ filtration.}$$

Note that for real-valued X_i with $X_0 = 0$ and sums $S_n = \sum_{i=1}^n X_i$ we have

$$\mathcal{F}_t = \sigma(X_0, \ldots X_t) = \sigma(S_0, \ldots S_t).$$

Conditional expectation as a random variable. [from Lecture 4]

Given r.v.'s (W, Y) consider $\mathbb{E}(W|Y = y)$. This is a number depending on y – in other words it's a **function** of y. Giving this function a name h we have

$$(*) \quad \mathbb{E}(W|Y = y) = h(y) \text{ for all possible values } y \text{ of } Y.$$

We now make a notational convention, to rewrite the assertion (*) as

$$(**) \quad \mathbb{E}(W|Y) = h(Y).$$

The right side is a r.v., so we must regard $\mathbb{E}(W|Y)$ as a r.v.

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Let's relate this to today's lecture using an elementary example. Take X and Y independent die throws

$$\mathbb{E}(X + Y|X) = X + \tfrac{7}{2}.$$

Consider $X^* = 10X$; then

$$\mathbb{E}(X + Y|X^*) = X + \tfrac{7}{2}.$$

because knowing the value of $10X$ is the same "information" as knowing the value of X.

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The central abstract idea of this lecture

It makes sense to talk about $\mathbb{E}(W|\mathcal{F})$ for a σ-field $\mathcal{F}$.

In the case $\mathcal{F} = \sigma(Y)$ we have $\mathbb{E}(W|\mathcal{F}) = \mathbb{E}(W|Y)$, as defined above. In general $\mathbb{E}(W|\mathcal{F})$ is a $\mathcal{F}$-measurable RV.

Recall the gambling interpretation of expectation

- A fair bet (really, the fair odds for a bet) is one where your gain G has $\mathbb{E}G = 0$.
- To receive a random amount of money X tomorrow, a fair "stake" to pay today is $\mathbb{E}X$

Now suppose we have some relevant "information" $\mathcal{F}$. The fair stake (to receive X) may depend on the information; this makes the fair stake a RV Z which must be $\mathcal{F}$-measurable. Consider the gambling strategy: choose an event A in $\mathcal{F}$

place the bet if A happens; don't bet if A does not happen.

Our gain is $G = (X - Z)1_A$, and for the stake to be fair we must have $\mathbb{E}G = 0$. This argument leads to two related ideas.

- The **gambling interpretation** of $\mathbb{E}(X|\mathcal{F})$ is as the fair stake Z to pay today in order to receive X tomorrow, when $\mathcal{F}$ is the known information.
- The **abstract math definition** of $\mathbb{E}(X|\mathcal{F})$ is as the $\mathcal{F}$-measurable RV Z such that

$$\mathbb{E}[Z1_A] = \mathbb{E}[X1_A] \text{ for all } A \text{ in } \mathcal{F}.$$

We will study martingales. A martingale is a real-valued stochastic process $(X_0, X_1, X_2, \ldots)$ with a certain property. Thinking of X_t as your "fortune" (amount of money) at time t, the property is

- X_t is your time-t fortune in some sequence of fair bets.

The general math definition is as follows.

- There is a filtration

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$$

($\mathcal{F}_t$ is the σ-field of known events at time t)

- The process $(X_0, X_1, X_2, \dots)$ is **adapted** to the filtration, meaning X_t is $\mathcal{F}_t$-measurable – we know the value of X_t at time t.
- $\mathbb{E}(X_{t+1}|\mathcal{F}_t) = X_t$ for each $t \geq 0$.

When this holds, (X_t) is a **martingale**. If the filtration is not specified then we take the natural filtration $\mathcal{F}_t = \sigma(X_0, \dots, X_t)$.

Why study martingales? 3 reasons

- Financial activities involving risk (stocks, insurance) are mathematically rather like gambling.
- One can often find aspects of other stochastic processes that are like martingales, so the theory (like calculus or algebra) is mathematically useful quite widely within Probability Theory.
- For any real-world future event A, the probabilities X_t that the event happens, given what is known at time t, must be a martingale.