Lecture 23

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Renewal processes. I will talk about only a very little of the material in [PK] Chapter 7.

Mental picture: light bulbs have random lifetime X, and we replace at failure. So successive bulbs have IID lifetimes $X_1, X_2, X_3, \ldots$ and we can consider

- $W_n = \sum_{i=1}^n X_i$ = time of i'th **renewal**
- $N(t) = \max\{n : W_n \leq t\}$ = number of renewals before t.

If X has Exponential(λ) distribution then the renewals form a rate-λ Poisson point process, but here we allow a general distribution for X. Write $\mu = \mathbb{E}X$. The law of large numbers says that as $n \to \infty$

$$W_n/n \to \mu \text{ a.s.}, \quad \mathbb{E}W_n/n \to \mu.$$

It is intuitively clear that we can rewrite this "upside down": on average we must replace a bulb every μ time units, that is at average rate $1/\mu$ per unit time, so

$$N(t)/t \to 1/\mu \text{ a.s.}, \quad \mathbb{E}N(t)/t \to 1/\mu.$$

We can rewrite this in terms of the **rate** of renewals at t. That is, defining

$$\lambda(t)\ dt = \mathbb{P}(\text{ some renewal in } [t, t+dt])$$

we have $\lambda(t) \to 1/\mu$ as $t \to \infty$.

Here is the first "interesting" result about renewal processes. The following are defined relative to a time t [board]

- $\delta_t = t - W_{N(t)}$ = time since last renewal before t
- $\gamma_t = W_{N(t)+1} - t$ = time until first renewal after t
- $\beta_t = \delta_t + \gamma_t$ = length of inter-renewal interval containing t.

Recall $\mu = \mathbb{E}X$ and write $F(x) = \mathbb{P}(X \le x)$ and $f_X(x)$ for its density function.

Theorem

As $t \to \infty$ the joint distribution (δ_t, γ_t) converges to the distribution of (δ, γ) defined by the joint density

$$f_{\delta,\gamma}(a, c) = f_X(a+c)/\mu.$$

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From this we can work out the marginal density of γ

$$f_\gamma(c) = \int_0^\infty f_{\delta,\gamma}(a, c)da = (1 - F(c))/\mu, \ 0 < c < \infty.$$

For δ we get the same result

$$f_\delta(a) = (1 - F(a))/\mu, \ 0 < a < \infty.$$

For β we get

$$f_\beta(b) = \int f_{\delta,\gamma}(a, b - a)da = bf_X(b)/\mu, \ 0 < b < \infty.$$

This is the "size-biased" distribution arising from X, discussed in a different setting in Lecture 2. [next slide]

- $X =$ number of children in a uniform random family
- $\widetilde{X} =$number of children in the family of a uniform randomly picked child.
- $N =$ number of families.

We calculate

- Number of families with i children $= N\mathbb{P}(X = i)$
- Number of children in i-child families $= i \times N\mathbb{P}(X = i)$
- Total number of children $= \sum_i i \times N\mathbb{P}(X = i) = N\, \mathbb{E}X$

$$\mathbb{P}(\widetilde{X} = i) = \frac{i \times N\mathbb{P}(X = i)}{N\, \mathbb{E}X} = \frac{i\, \mathbb{P}(X = i)}{\mathbb{E}X}.$$

Say $\widetilde{X}$ has the **size-biased** distribution of X.

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In the light bulb (renewal theory) setting this is an explanation of the **inspection paradox;** the mean total lifetime $\mathbb{E}\beta$ of the bulb in use at a given time is larger than the mean lifetime $\mathbb{E}X$ of a typical bulb.

The **cycle trick** mentioned in previous lectures is part of renewal theory. Suppose there are IID rewards R_i associated with renewals – precisely, the pairs $(X_1, R_1), (X_2, R_2), \ldots$ are IID. Then

$$\text{long-run average reward per unit time} = \mathbb{E}R/\mathbb{E}X.$$

Example: scheduling replacements before failure.
In many examples other than light bulbs (e.g. car battery), the cost C_1 of replacement before failure is less than the cost C_2 of replacement at failure. So we can consider a policy:

replace at (random) failure time X or at (fixed) time T, whichever comes first.

What is the optimal choice of T?

- Replace at time $X^* = \min(X, T)$
- Incur cost $C = C_2$ if $X < T$, or cost $C = C_1$ if $X > T$.

Long-run average cost per unit time $= \mathbb{E}C/\mathbb{E}X^*$.

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We need to write these quantities in terms of $F(x) = \mathbb{P}(X \leq x)$ and the associated density function $f(x)$.

$$\mathbb{E}X^* = \int_0^T xf(x)dx + T(1 - F(T)).$$

$$\mathbb{E}C = C_2F(T) + C_1(1 - F(T)).$$

We could now use calculus or numerics to find the value of T which minimizes $\mathbb{E}C/\mathbb{E}X^*$.

The IID central limit theorem (CLT) says

$$\frac{W_n - n\mu}{\sigma n^{1/2}} \to_d \mathrm{Normal}(0,1)$$

$$\mathbb{P}(W_n > t) \approx 1 - \Phi(\frac{t - n\mu}{\sigma n^{1/2}}).$$

We expect a corresponding CLT for the renewal counting process $N(t)$: for some "unknown" q

$$\frac{N(t) - t/\mu}{qt^{1/2}} \to_d \mathrm{Normal}(0,1) \quad ???$$

$$\mathbb{P}(N(t) < n) \approx \Phi(\frac{n - t/\mu}{qt^{1/2}}) \quad ???$$

But the events $\{W_n > t\}$ and $\{N(t) < n\}$ are the same, and this enables us to calculate q [board] by considering n and t related by

$$n = t/\mu + qt^{1/2}z.$$

So we get

$$q = \sigma/\mu^{3/2}$$

and then indeed

$$\frac{N(t) - t/\mu}{qt^{1/2}} \to_d \mathrm{Normal}(0,1)$$

$$\mathbb{P}(N(t) \leq n) \approx \Phi(\frac{n - t/\mu}{qt^{1/2}}).$$