# Lecture 22

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In continuous time $0 \leq t < \infty$ we specify transition **rates**

$$q_{ij} = \lim_{\delta \downarrow 0} \frac{\mathbb{P}(X(t+\delta)=j|X(t)=i, \text{ past })}{\delta}$$

or informally

$$\mathbb{P}(X(t+dt) = j|X(t) = i) = q_{ij}dt$$

but note these are defined only for $j \neq i$. The time-$t$ distribution $\pi(t)$ evolves as

$$\frac{d}{dt}\pi(t) = \pi(t)\mathbf{Q}$$

where $\mathbf{Q}$ is the matrix with off-diagonal entries $(q_{ij})$ and with diagonal entries defined by

$$q_{ii} = -q_i = -\sum_{j \neq i} q_{ij}.$$

**Birth-and-death chains.**

These have states $\{0, 1, 2, \ldots, N\}$ or $\{0, 1, 2, \ldots\ldots\}$ and the only transitions are $i \to i \pm 1$. Write

$$\lambda_i = q_{i,i+1} \ \text{(birth rate)}; \qquad \mu_i = q_{i,i-1} \ \text{(death rate)}.$$

For these chains we can solve the detailed balance equations: [board]

$$w_i = \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j}; \quad w_0 = 1, \ w = \sum_{i \geq 0} w_i.$$

So the stationary distribution is

$$\pi_i = w_i / w$$

provided (in the infinite-state case) $w < \infty$.

**Example.** Take $\lambda_i = \lambda, \ \mu_i = \mu i$. Then [board] $\pi$ is the Poisson$(\lambda/\mu)$ distribution.

**Example.** Take $\lambda_i = \lambda, \ \mu_i = \mu, \ \lambda < \mu$. Then [board] $\pi$ is the shifted Geometric ($p = 1 - \lambda/\mu$) distribution.

This is the M/M/1 queue model, as follows.

- Customers arrive at times of a rate-$\lambda$ Poisson point process
- Service times are IID Exponential($\mu$).
- $X(t)$ = number of customers at time $t$,

We can calculate many quantities associated with the stationary process [board]

- Long-run proportion of time server is idle $= 1 - \lambda/\mu$.
- Mean number of customers $= \frac{\lambda}{\mu-\lambda}$.
- Mean waiting time (until starting service) for customer $= \frac{\lambda/\mu}{\mu-\lambda}$.
- Mean total time (until ending service) for customer $= \frac{1}{\mu-\lambda}$.
- Mean busy period for server $= \frac{1}{\mu-\lambda}$.

**More theory – similar to discrete-time setting.**

[Assume chain is irreducible, and either finite-state or infinite state and positive-recurrent, so a unique stationary distribution $\pi$ exists.]

- For any initial distribution, $\mathbb{P}(X(t) = i) \to \pi_i$ as $t \to \infty$.
- Writing $N_i(t) =$ length of time chain spends in state $i$ during $[0, t]$, we have $N_i(t)/t \to \pi_i$ as $t \to \infty$.
- $\mathbb{E}_i T_i^+ = 1/(\pi_i q_i)$, where $T_i^+$ is the first **return time** to $i$ (after leaving $i$).

Note we don't need "aperiodic" in the first result. The third result can be seen by a general "cycle argument" [next slide and board].

**Example: repairman model** ([PK] Problem 6.4.3.)

- 5 machines – each is working or "failing" (not working)
- A working machine fails at rate $\alpha = 0.2$
- 1 repairman; a repair takes random Exponential(rate $\beta = 0.5$) time
- Study $X(t)$ =number of machines working at time $t$.

[board]

Note that if the stationary distribution $\pi$ exists for an infinite-state birth-and-death process, then for the same process on states $\{0, 1, 2, \ldots, N\}$ the stationary distribution is

$$\pi_i^{[N]} = \pi_i / s. \quad s = \sum_{j=0}^{N} \pi_j.$$

In other words, taking $\pi$ as the distribution of a RV $Z$, $\pi^{[N]}$ is the conditional distribution of $Z$ given $\{Z \leq N\}$.