Lecture 20

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Continuous-time Markov chains [PK] section 6.6

In discrete time $t = 0, 1, 2, \ldots$ we specify a Markov chain by specifying the matrix **P** of transition probabilities

$$p_{ij} = \mathbb{P}(X(t+1) = j | X(t) = i, \text{ past }).$$

In continuous time $0 \le t < \infty$ we specify transition **rates**

$$q_{ij} = \lim_{\delta \downarrow 0} \frac{\mathbb{P}(X(t+\delta) = j | X(t) = i, \text{ past })}{\delta}$$

or informally

$$\mathbb{P}(X(t+dt) = j | X(t) = i) = q_{ij} dt$$

but note these are defined only for $j \neq i$. Then note that

$$\begin{aligned} \mathbb{P}(X(t+dt) \neq i | X(t) = i) &= \sum_{j \neq i} \mathbb{P}(X(t+dt) = j | X(t) = i) \\ &= \sum_{j \neq i} q_{ij} dt = q_i dt \end{aligned}$$

where

$$q_i = \sum_{j \neq i} q_{ij}.$$

In discrete time the time-t distribution $\pi(t) = (\pi_i(t)) = (\mathbb{P}(X(t) = i))$ evolves as $\pi(t+1) = \pi(t)\mathbf{P}$. In continuous time we have [board]

$$\tfrac{d}{dt}\pi_j(t) = \sum_{i \neq j} \pi_i(t) q_{ij} - \pi_j(t) q_j \qquad q_j := \sum_{k \neq j} q_{jk}.$$

We can re-write this in vector-matrix notation as

$$\tfrac{d}{dt}\pi(t) = \pi(t)\mathbf{Q}$$

where $\mathbf{Q}$ is the matrix with off-diagonal entries (q_{ij}) and with diagonal entries defined by

$$q_{ii} = -q_i = -\sum_{j \neq i} q_{ij}.$$

Note this implies that the condition for a probability distribution π to be a stationary distribution is

$$\pi\mathbf{Q} = 0 \quad \text{(the zero vector).}$$

Note [PK] write $\mathbf{A}$ instead of $\mathbf{Q}$.

Starting at state i,

$$S_i = \min\{t : X(t) \neq i\}$$

is called the **sojourn time** in i. It is the time spent at i before jumping to another state. The fact

$$\mathbb{P}(X(t+dt) \neq i | X(t) = i) = q_i dt$$

is the fact

$$\mathbb{P}(S_i \in [t, t+dt] \,|S_i > t) = q_i dt$$

which shows that S_i has Exponential(q_i) distribution. At time S_i the process jumps to another state: the probability it jumps to state j is [board]

$$\widehat{p}_{ij} = q_{ij}/q_i.$$

This leads to a "jump and hold" description of a continuous-time Markov chain.

- After jumping into a state i, the process remains in state i for a random time with Exponential(q_i) distribution.
- Then it jumps to some other state, to state $j \neq i$ with probability $\widehat{p}_{ij} = q_{ij}/q_i$.

So the matrix

$$\widehat{\mathbf{P}} = (\widehat{p}_{ij}), \quad \text{where } \widehat{p}_{ii} = 0$$

is the transition matrix for the discrete-time **jump chain** $\hat{X}(0), \hat{X}(1), \ldots$ that shows the successive states visited.

The relationship between the stationary distributions (where they exist) π and $\widehat{\pi}$ can be seen using a long-run argument [board] or algebraically from the equations $\widehat{\pi}\widehat{\mathbf{P}} = \widehat{\pi}, \quad \pi\mathbf{Q} = 0$:

$$\pi_i = c\widehat{\pi}_i/q_i; \quad \widehat{\pi}_i = c^{-1}q_i\pi_i$$

for $c = \frac{1}{\sum_j \widehat{\pi}_i/q_i} = \sum_j q_i\pi_i$.

In very special cases we can solve the differential equations

$$\frac{d}{dt}\pi(t) = \pi(t)\mathbf{Q}$$

where $\mathbf{Q}$ is the matrix with off-diagonal entries (q_{ij}) and with diagonal entries defined by

$$q_{ii} = -q_i = -\sum_{j \neq i} q_{ij}.$$

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Example: For the rate-λ PPP on $[0, \infty)$ the counting process $N(t)$ is the continuous-time chain with $q_{i,i+1} = \lambda$.

Example: 2-state chain: $q_{01} = \lambda, \ q_{10} = \mu$.
[board]

$$\mathbb{P}_0(X(t) = 0) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \exp(-(\lambda+\mu)t).$$

Example: Yule process

- parameter $\beta > 0$
- states $1, 2, 3, \ldots$
- transition rates $q_{i,i+1} = \beta i$
- $X(0) = 1$.

The differential equations are

$$\frac{d}{dt}\pi_j(t) = \beta[(j-1)\pi_{j-1}(t) - j\pi_j(t)].$$

One can solve these equations – see [PK] section 6.1.3

$$\pi_j(t) = \mathbb{P}(X(t) = j) = e^{-\beta t}(1 - e^{-\beta t})^{j-1}, \ j = 1, 2, \ldots$$

In other words $X(t)$ has Geometric $e^{-\beta t}$ distribution, so $\mathbb{E}X(t) = e^{\beta t}$.

The Yule process is a basic example of a continuous-time branching process [picture on board]

The Yule process is also an example of a "pure birth" process, meaning the only transitions are $i \to i+1$. For such processes the distribution of $X(t)$ can be related to the sum of independent Exponentials RVs – see [PK] section 6.1.2.