Lecture 18

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9 October 2015

Lemma

If (ξ_i) are the points of a rate-1 PPP on $[0, \infty)$ and if $G : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing, with $g(x) = \frac{dG}{dx}$ and $G(0) = 0$, then the points $(G(\xi_i))$ form another a PPP on $[0, \infty)$ with rate $\lambda(y) = 1/g(G^{-1}(y))$.

This allows us to construct (mathematically) the PPP with rate function $\lambda(t)$ by solving (for G) the equation

$$\lambda(y) = 1/g(G^{-1}(y)).$$

More usefully, because we know how to simulate the rate-1 PPP on $[0, \infty)$ (inter-event times are IID Exponential(1)) this enables us to **simulate** the PPP with rate function $\lambda(t)$.
As an example, consider $G(x) = ax^{1/2}$. Then [board] $\lambda(y) = 2y/a^2$.

Recall that the distances $D_1, D_2, D_3 \ldots$ to the origin in a 2-dimensional rate$-\lambda$ PPP are the points of a PPP on $[0, \infty)$ with rate function $\lambda(r) = 2\pi\lambda r$. So we can simulate $D_1, D_2, D_3 \ldots$ using $G(x) = ax^{1/2}$ with $a = (\pi\lambda)^{-1/2}$.

[from earlier class, for constant-rate PPP on $[0, \infty)$]

Theorem

Fix $t > 0$ and $k \geq 1$. Conditional on $\{N(t) = k\}$ the times $(W_1, W_2, \ldots, W_k)$ of events in the PPP are distributed as the order statistics of k IID Uniform$(0, t)$ random variables.

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The analogous result holds in two dimensions.

Theorem

Let $B \subset \mathbb{R}^2$ be a region with finite area. Then the rate-λ PPP on B can be constructed as follows.
(i) Take $N(B)$ with Poisson$(\lambda \times \mathrm{area}(B))$ distribution.
(ii) Given $N(B) = n$, take n random points independent uniform on B.

Question: How could we simulate a rate-λ PPP on some region B in $\mathbb{R}^2$.

Not obvious how to simulate, but using theory we see two ways.

(1) If B is a square then we could use Theorem above. Easy to sample uniformly from square in (x,y)- coordinates.

(2) If B is a disc then we know how to simulate the radial distances $D_1, D_2, D_3, \ldots$. So use polar coordinates (D_i, θ_i); intuitively clear the θ_i are IID uniform on $(0, 2\pi)$.

Spatial PPP with varying rate $\lambda(x, y)$.

- $N(A)$ has Poisson$(\int_A \lambda(x, y)dxdy)$ distribution.
- $\mathbb{P}($ some point in $[x, x + dx) \times [y, y + dy]$ $) = \lambda(x, y)dxdy$.
- For disjoint $A_1, A_2, \ldots$ the random variables $N(A_i)$ are independent.

An interesting use of this idea is to combine time with space, as follows. Suppose we have a rate-λ PPP of times of events $0 < W_1 < W_2 < \ldots$. Suppose that associated with the i'th event is a $\mathbb{R}$-valued random variable Y_i, where $(Y_1, Y_2, \ldots)$ are IID with density $g(y)$, independent of (W_i). Then we can regard the points (W_1, Y_1), (W_2, Y_2), $\ldots$ as a point process on $[0, \infty) \times (-\infty, \infty)$.

Theorem (PK Theorem 5.8)

The points (W_1, Y_1), (W_2, Y_2), $\ldots$ form a Poisson PP with rate $\lambda(t, y) = \lambda g(y)$.

[KP] Exercise 5.6.10.

- You want to sell at item before time 1.
- Bids arrive at times of a rate-1 PPP; you must accept/reject bis at that time.
- Bid amounts $U_1, U_2, \ldots$ are IID Uniform $[0, 1]$.

What is a good strategy?

Strategy A; Fix a price θ and accept first bid over θ.

Analysis [board]:

$$\mathbb{E}(\text{ price received}) = \frac{1+\theta}{2}\ (1 - e^{-(1-\theta)}).$$

Intuition suggests it would be better to use a decreasing threshold for accepting a bid.

Strategy B; Accept the first bid which (at time t) is larger than $\theta(t) = \frac{1-t}{3-t}$.

[details on board: outline here].

(a) Points (T_i, U_i) are PPP on $[0,1] \times [0,1]$ of rate $\lambda(t,u) = \lambda g_U(u) = 1$.

(b) Consider
$g(t,u)dtdu = \mathbb{P}($ bid offered and accepted in $[t, t+dt] \times [u, u+du])$.
After a calculation, $g(t,u) = (1 - \frac{t}{3})^2$.

(c)

$$\mathbb{E}(\text{ price received}) = \int \int_{D_1} u \; g(t,u) \; dtdu.$$