# Lecture 17

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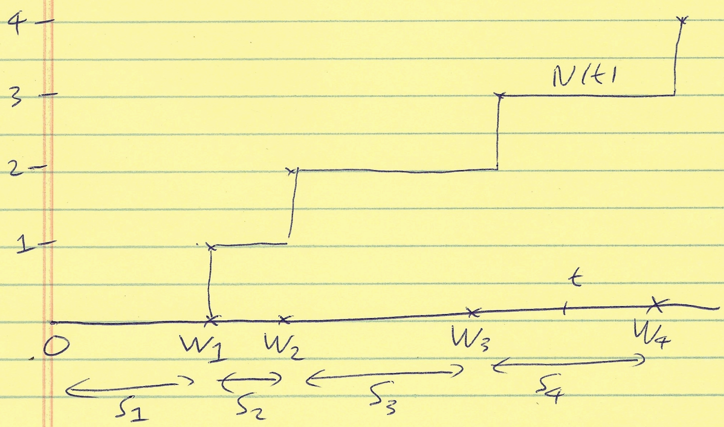
## Ideas from previous lecture.

Notation for studying random times (of events) over $0 \le t < \infty$. Cannot have two events at the same time.

- $W_k$ = time at which $k$'th event occurs ($W_0 = 0$).
- $S_k = W_k - W_{k-1}$ is the time between successive events.
- $N(t)$ = number of events during time $[0, t]$.
- $N(s, t) = N(t) - N(s)$ = number of events during $(s, t]$.

Note that the event $\{W_n \le t\}$ is the same as the event $\{N(t) \ge n\}$. So, regardless of the probability model,

$$\mathbb{P}(W_n \le t) = \mathbb{P}(N(t) \ge n).$$


4
3
2
1
N(t)
t
0
$W_1$
$W_2$
$W_3$
$W_4$
$S_1$
$S_2$
$S_3$
$S_4$

## Poisson point process (PPP) of rate $\lambda$.

This process is defined by the properties
(a) $N(s, t)$ has Poisson($\lambda(t - s)$) distribution.
(b) For disjoint intervals $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ the random variables $N(s_1, t_1), N(s_2, t_2), \ldots, N(s_k, t_k)$ are independent.

A more informal description in terms of infinitesimal intervals is
(a') $\mathbb{P}($ event during $[t, t + dt]) = \lambda\ dt$
(b') What happens in disjoint time intervals is independent.

The PPP is used as an over-simplified model for events that occur at "completely random" times

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**Theorem**

*Given two independent PPPs with rates $\lambda_1$ and $\lambda_2$, the combined process – that is the process with $N(t) = N_1(t) + N_2(t)$ – is a PPP with rate $\lambda_1 + \lambda_2$.*

**Theorem**

*Let $p_1, p_2, \ldots$ be a probability distribution on "colors" $1, 2, 3, \ldots$. Take a rate-$\lambda$ PPP, and assign colors to the points, each point independently getting color $i$ with probability $p_i$. Then*
*(i) For each $i$ the process of color-$i$ points is a PPP with rate $\lambda p_i$.*
*(ii) These processes are independent as $i$ varies.*

For instance, if we model times of "accidents" as a rate-$\lambda$ PPP, and then model each accident as "serious" with probability $p$ and "not serious" with probability $1 - p$, then

- serious accidents occur as a PPP of rate $\lambda p$
- non-serious accidents occur as a PPP of rate $\lambda(1 - p)$
- the two processes are independent.

"Independence" here looks intuitively wrong but arises from the assumption that $\lambda$ is known.

## Spatial Poisson processes

We have been imagining the line $[0, \infty)$ as "time", but the mathematics is the same for 1-dimensional space – for instance, positions of accidents along a highway.

More interesting to consider random points in 2-dimensional space. Here the rate $\lambda$ will be the mean number of points per unit area. Write $N(A)$ for the number of points in a region $A$. The rate-$\lambda$ PPP on $\mathbb{R}^2$ is defined by the properties

- $N(A)$ has Poisson$(\lambda \times \mathrm{area}(A))$ distribution.
- For disjoint $A_1, A_2, \ldots$ the random variables $N(A_i)$ are independent.

This is used as a model of "purely random" points.

In a rate-$\lambda$ PPP on $\mathbb{R}^2$, let $D_k$ be the distance from the origin to the $k$'th closest point.

Here are some results about this [board].

$D_1$ has density $f_{D_1}(x) = 2\pi\lambda x \exp(-\pi\lambda x^2)$.

$D_1, D_2, D_3 \ldots$ are the points of a PPP on $[0, \infty)$ with rate function $\lambda(r) = 2\pi\lambda r$.

## The PPP with rate function $\lambda(t)$.

Start with the informal description in terms of infinitesimal intervals:
(a') $\mathbb{P}($ event during $[t, t + dt]) = \lambda(t)\ dt$
(b') What happens in disjoint time intervals is independent.

We can then deduce the other description. Write $\Lambda(t) = \int_0^t \lambda(u)\ du$.

(a) $N(s, t)$ has Poisson$(\Lambda(t) - \Lambda(s))$ distribution.
(b) For disjoint intervals $(s_1, t_1), \ (s_2, t_2), \ \ldots, (s_k, t_k)$ the random variables $N(s_1, t_1), \ N(s_2, t_2), \ \ldots, \ N(s_k, t_k)$ are independent.

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We can explain this via a general result and then a special result. [board]

**Lemma**

*If $(\xi_i)$ are the points of a PPP (in time or space, and maybe varying rate) then for any function $G$ the points $(G(\xi_i))$ form another PPP (typically with varying rate).*

**Lemma**

*If $(\xi_i)$ are the points of a rate-1 PPP on $[0, \infty)$ and if $G : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing, with $g(x) = \frac{dG}{dx}$ and $G(0) = 0$, then the points $(G(\xi_i))$ form another a PPP on $[0, \infty)$ with rate $\lambda(y) = 1/g(G^{-1}(y))$.*