Lecture 16

David Aldous

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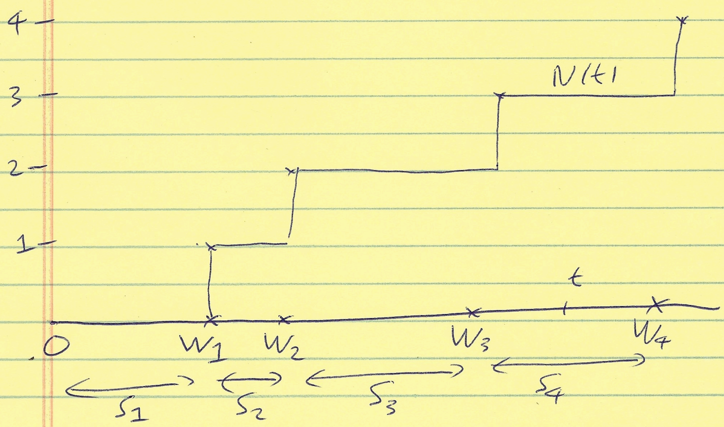
Notation for studying random times (of events) over $0 \le t < \infty$. Cannot have two events at the same time.

- W_k = time at which k'th event occurs ($W_0 = 0$).
- $S_k = W_k - W_{k-1}$ is the time between successive events.
- $N(t)$ = number of events during time $[0, t]$.
- $N(s, t) = N(t) - N(s)$ = number of events during $(s, t]$.

Note that the event $\{W_n \le t\}$ is the same as the event $\{N(t) \ge n\}$. So, regardless of the probability model,

$$\mathbb{P}(W_n \le t) = \mathbb{P}(N(t) \ge n).$$

We will study the mathematically simplest probability model. Fix a parameter $0 < \lambda < \infty$.



4
3
2
1
N(t)
t
0
W_1
W_2
W_3
W_4
S_1
S_2
S_3
S_4

Poisson point process (PPP) of rate λ.

This process is defined by the properties
(a) $N(s,t)$ has Poisson($\lambda(t-s)$) distribution.
(b) For disjoint intervals $(s_1,t_1),\ (s_2,t_2),\ \ldots,(s_k,t_k)$ the random variables $N(s_1,t_1),\ N(s_2,t_2),\ \ldots,\ N(s_k,t_k)$ are independent.

Why does such a process exist? Consider large M, and suppose events could only happen at times $\frac{1}{M},\frac{2}{M},\frac{3}{M},\ldots$, independently with probability λ/M. Then the number of events during (s,t) is almost Binomial($M(t-s),\lambda/M$). We picture the PPP as the $M\to\infty$ limit, and the Poisson distribution arises as limit of Binomials.

A more informal description in terms of infinitesimal intervals is
(a') $\mathbb{P}($ event during $[t,t+dt]) = \lambda\ dt$
(b') What happens in disjoint time intervals is independent.

Conceptual point; "Poisson" is not an arbitrary assumption, but instead arises automatically from (a',b'). Also λ is a "rate": the mean number of events per unit time.

The PPP is used as an over-simplified model for events that occur at "completely random" times

- accidents
- coincidences
- start of phone calls (from phone company viewpoint)
- customer joining supermarket checkout line (from the supermarket viewpoint)
- earthquakes
- murders

For many of these examples we know that in fact the rates vary with time. But we can adapt the model to allow a time-varying rate function $\lambda(t)$.

The PPP with rate function $\lambda(t)$.

Start with the informal description in terms of infinitesimal intervals:
(a') $\mathbb{P}($ event during $[t, t+dt]) = \lambda(t)\ dt$
(b') What happens in disjoint time intervals is independent.

We can then deduce the other description. Write $\Lambda(t) = \int_0^t \lambda(u)\ du$.

(a) $N(s,t)$ has Poisson$(\Lambda(t) - \Lambda(s))$ distribution.
(b) For disjoint intervals $(s_1, t_1),\ (s_2, t_2),\ \ldots, (s_k, t_k)$ the random variables $N(s_1, t_1),\ N(s_2, t_2),\ \ldots,\ N(s_k, t_k)$ are independent.

We will see details later. For now, the **mathematical** point is that we can deduce results for the "rate function $\lambda(t)$" case from results for the "constant rate λ" case, so we can set up theory in the constant rate case. A **modeling** point is that any process of random events has some mean rate function $\lambda(t)$ – what is special about the Poisson process is the independence property. Is this approximately true in a given example?

Poisson point process (PPP) of rate λ defined by the properties

(a) $N(s,t)$ has Poisson$(\lambda(t-s))$ distribution.
(b) For disjoint intervals $(s_1,t_1),\ (s_2,t_2),\ \ldots,(s_k,t_k)$ the random variables $N(s_1,t_1),\ N(s_2,t_2),\ \ldots,\ N(s_k,t_k)$ are independent.

In this model we will consider

- W_k = time at which k'th event occurs ($W_0 = 0$).
- $S_k = W_k - W_{k-1}$ is the time between successive events.

- -

The first results are [board]

- W_1 has Exponential(λ) distribution
- W_k has Gamma(λ, k) distribution:

$$f_{W_k}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}.$$

- $S_1, S_2, S_3, \ldots$ are IID Exponential(λ).

Note that we can use the fact

$$S_1, S_2, S_3, \ldots \text{ are IID Exponential}(\lambda)$$

as a (mathematical) construction of the PPP, or as an easy way to simulate the process.

Here is the next result.

Theorem

Fix $t > 0$ and $k \geq 1$. Conditional on $\{N(t) = k\}$ the times $(W_1, W_2, \ldots, W_k)$ of events in the PPP are distributed as the order statistics of k IID Uniform$(0, t)$ random variables.

[board]

The same result holds if instead we condition on $\{W_{k+1} = t\}$.