Events A and B are **independent** if:
knowing whether A occured does not change the probability of B.

Mathematically, can say in two equivalent ways:

$$\begin{aligned} P(B|A) &= P(B) \\ P(A \text{ and } B) = P(B \cap A) &= P(B) \times P(A). \end{aligned}$$

Important to distinguish independence from **mutually exclusive** which would say $B \cap A$ is empty (cannot happen).

Example. Deal 2 cards from deck

A	first card is Ace
C	second card is Ace

$$P(C|A) = \frac{3}{51}$$

$$P(C) = \frac{4}{52} \text{(last class)}.$$

So A and C are **dependent**.

Example. Throw 2 dice

A first die lands 1

B second die shows larger number than first die

C both dice show same number

$$P(B|A) = \tfrac{5}{6} \quad P(B) =? = \tfrac{15}{36} \text{ by counting}$$

so A and B **dependent**.

$$P(C|A) = \tfrac{1}{6} \quad P(C) = \tfrac{6}{36} = \tfrac{1}{6}$$

so A and C **independent**.
Note 1: here B and C are mutually exclusive.
Note 2: writing $B' =$ "second die shows smaller number than first die" we have

$$P(B') = P(B) \text{ by symmetry}$$

$$P(B \cup B') = P(C^c) = 1 - P(C) = \tfrac{5}{6}$$

giving a "non-counting" argument that $P(B) = \frac{5}{12}$.

Example. Deal 1 card from deck

A card is Ace
S card is Spade

$$P(A) = \frac{4}{52} \quad P(S) = \frac{13}{52} \quad P(A \cap S) = \frac{1}{52}.$$

Here $P(A \cap S) = P(A) \times P(S)$ so **independent**.

Conceptual point.
(a) In a fully-specified math model, two events are either dependent or independent; can be checked by calculation.
(b) Often we use independence as an **assumption** in making a model. For instance we **assume** that different die throws give independent results. Most probability models one encounters in engineering or science have some assumption of "bottom level" independence; but one needs to be careful about which other events within the model are independent.

(silly) Example.
Throw 2 dice. If sum is at least 7 I show you the dice; if not, I don't.

A: I show you first die lands 1
B: I show you second die lands 1

$$P(A) = \frac{1}{36}, \quad P(B) = \frac{1}{36}, \quad P(A \cap B) = 0$$

so A and B **dependent**.

Conceptual point. This illustrates a subtle point: being told by a truthful person that "A happened" is not (for probability/statistics purposes) exactly the same as "knowing A happened".

[car accident example]

Systems of components

Will show **logic diagrams**: system works if there is some path left-to-right which passes only though working components.

Assume components work/fail independently,

$$P(C_i \text{ works }) = p_i, \quad P(C_i \text{ fails }) = 1 - p_i.$$

Note in practice the independence assumption is usually unrealistic.

Math question: calculate $P($ system works $)$ in terms of the numbers p_i and the network structure.

Example: "in series".

[picture on board]

$$P(\text{ system works }) = p_1 p_2 p_3.$$

Example: "in parallel".

[picture on board]

$$P(\text{ system fails }) = (1 - p_1)(1 - p_2)(1 - p_3).$$

$$P(\text{ system works }) = 1 - (1 - p_1)(1 - p_2)(1 - p_3).$$

More complicated example:

[picture on board]

We could write out all 16 combinations; instead let's condition on whether or not C_1 works.

$$
\begin{aligned}
P(\text{system works}) &= P(\text{system works} \mid C_1 \text{ works}) P(C_1 \text{ works}) \\
&+ P(\text{system works} \mid C_1 \text{ fails}) P(C_1 \text{ fails})
\end{aligned}
$$

[continue on board]

Example: Deal 4 cards. What is chance we get exactly one Spade?

event	1st	2nd	3rd	4th
F_1	S	N	N	N
F_2	N	S	N	N
F_3				
F_4	N	N	N	S

[board: repeated conditioning]

$$P(F_1) = \frac{13}{52} \times \frac{39}{51} \times \frac{38}{50} \times \frac{37}{49}$$

$$P(F_1) = P(F_2) = P(F_3) = P(F_4)$$

$$P(\text{exactly one Spade}) = P(F_1 \text{ or } F_2 \text{ or } F_3 \text{ or } F_4))$$

$$= P(F_1) + P(F_2) + P(F_3) + P(F_4) = 4 \times P(F_1) \approx 44\%.$$

Example: Deal 4 cards. What is chance we get one card of each suit?

event	1st	2nd	3rd	4th
A_1	C	D	H	S
A_2	C	D	S	H
.	.	.	.	.
.	.	.	.	.

$$P(A_1) = \frac{13}{52} \times \frac{13}{51} \times \frac{13}{50} \times \frac{13}{49}$$

$$P(A_1) = P(A_2) = \ldots$$

Number of possible orders $= 4 \times 3 \times 2 \times 1 = 24 = 4!$

$$P(\text{one card of each suit}) = 24 \times P(A_1) \approx 10.5\%.$$

Bayes rule: updating probabilities as new information is acquired.
(silly) Example There are 2 coins:
one is fair: $P(\text{Heads}) = 1/2$; one is biased: $P(\text{Heads}) = 9/10$
Pick one coin at random. Toss 3 times. Suppose we get 3 Heads. What then is the chance that the coin we picked is the biased coin?

Abstract set-up: Partition $(B_1, B_2, \ldots)$ of "alternate possibilities".
Know **prior** probabilities $P(B_i)$.
Then observe some event A happens (the "new information") for which we know $P(A|B_i)$. We want to calculate the **posterior** probabilities $P(B_i|A)$.
Bayes formula:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots}.$$

[example above on board]