Spring 2013 Statistics 153 (Time Series) : Lecture Twenty Two

Aditya Guntuboyina

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1 Spectral Distribution Function

Let $\{X_t\}$ be a stationary sequence of random variables and let $\gamma_X(h) = \text{cov}(X_t, X_{t+h})$ denote the autocovariance function.

A theorem due to Herglotz (sometimes attributed to Bochner) states that **every** autocovariance function γ_X can be written as:

$$\gamma_X(h) = \int_{-1/2}^{1/2} e^{2\pi i h\lambda} dF(\lambda),$$

where $F(\cdot)$ is a non-negative, right-continuous, non-decreasing function on $[-1/2, 1/2]$ with $F(-1/2) = 0$ and $F(1/2) = \gamma_X(0)$. Moreover, F is uniquely determined by γ_X.

This function F is called the *Spectral Distribution Function* of $\{X_t\}$. If F has a density f i.e., if F can be written as

$$F(x) := \int_{-1/2}^{x} f(t)dt$$

then f is called the *Spectral Density* of $\{X_t\}$.

A sufficient condition (but not necessary) for the existence of the spectral density is the condition $\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty$. And in this case, the spectral density exists and is given by the formula:

$$f(\lambda) = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(-2\pi i \lambda h) \qquad \text{for } -1/2 \le \lambda \le 1/2.$$

The spectral distribution function is an as important quantity for a stationary process as the autocovariance function.

2 Linear Time-Invariant Filters

A linear time-invariant filter uses a set of specified coefficients $\{a_j\}$ for $j = \dots, -2, -1, 0, 1, 2, 3, \dots$ to transform an input time series $\{X_t\}$ into an output time series $\{Y_t\}$ according to the formula:

$$Y_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j}.$$

The filter is determined by the coefficients $\{a_j\}$ which are often assumed to satisfy $\sum_{j=-\infty}^{\infty} |a_j| < \infty$.

Suppose that the input series $\{X_t\}$ is given by

$$X_t = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Such an $\{X_t\}$ is often called an *impulse function*. The output of the filter $\{Y_t\}$ can then be easily seen to be $Y_t = a_t$. For this reason the filter coefficients $\{a_j\}$ are often collectively known as the *impulse response function*.

The two main examples of linear time-invariant filters that we have seen so far are (1) the moving average filter which has the impulse response function: $a_j = 1/(2q+1)$ for $|j| \leq q$ and $a_j = 0$ otherwise; and (2) Differencing which corresponds to the filter $a_0 = 1$ and $a_1 = -1$ and all other a_js equal zero. We have seen that these two filters act very differently; one estimates trend while the other eliminates it.

Suppose that the input time series $\{X_t\}$ is stationary with autocovariance function γ_X. What is the autocovariance function of $\{Y_t\}$? Observe that

$$\gamma_Y(h) := \text{cov}\left(\sum_j a_j X_{t-j}, \sum_k a_k X_{t+h-k}\right) = \sum_{j,k} a_j a_k \text{cov}(X_{t-j}, X_{t+h-k}) = \sum_{j,k} a_j a_k \gamma_X(h-k+j). \quad (1)$$

Note that the above calculation shows also that $\{Y_t\}$ is stationary.

Suppose now that the spectral density of the input stationary series $\{X_t\}$ is f_X. What then is the spectral density f_Y of the output $\{Y_t\}$?

Because the spectral density of $\{X_t\}$ equals f_X, we have

$$\gamma_X(h) = \int_{-1/2}^{1/2} e^{2\pi i h\lambda} f_X(\lambda) d\lambda.$$

We thus have from (1) that

$$\gamma_Y(h) = \sum_j \sum_k a_j a_k \int e^{2\pi i(h-k+j)\lambda} f_X(\lambda) d\lambda = \int e^{2\pi i h\lambda} f_X(\lambda) \left(\sum_j \sum_k a_j a_k e^{-2\pi i k\lambda} e^{2\pi i j\lambda}\right) d\lambda \quad (2)$$

Let us now define the function

$$A(\lambda) := \sum_j a_j e^{-2\pi i j\lambda} \qquad \text{for } -1/2 \leq \lambda \leq 1/2.$$

Note that this function only depends on the filter coefficients $\{a_j\}$. From (2) it clearly follows that

$$\gamma_Y(h) = \int e^{2\pi i\lambda h} f_X(\lambda) A(\lambda) \overline{A(\lambda)} d\lambda,$$

where, of course, $\overline{A(\lambda)}$ denotes the complex conjugate of $A(\lambda)$. As a result, we have

$$\gamma_Y(h) = \int e^{2\pi i\lambda h} f_X(\lambda) |A(\lambda)|^2 d\lambda.$$

This is clearly of the form $\gamma_Y(h) = \int e^{2\pi i\lambda h} f_Y(\lambda) d\lambda$. We therefore have

$$f_Y(\lambda) = f_X(\lambda) |A(\lambda)|^2 \qquad \text{for } -1/2 \leq \lambda \leq 1/2. \quad (3)$$

In other words, the action of the filter on the spectrum of the input is very easy to explain. It modifies the spectrum by multiplying it with the function $|A(\lambda)|^2$. Depending on the value of $|A(\lambda)|^2$, some frequencies may be enhanced in the output while other frequencies will be diminished.

This function $\lambda \mapsto |A(\lambda)|^2$ is called the *power transfer function* of the filter. The function $\lambda \mapsto A(\lambda)$ is called the *transfer function* or the *frequency response function* of the filter.

The spectral density is very useful while studying the properties of a filter. While the autocovariance function of the output series γ_Y depends in a complicated way on that of the input series γ_X, the dependence between the two spectral densities is very simple.

Example 2.1 (Power Transfer Function of the Differencing Filter)**.** *Consider the Lag s differencing filter: $Y_t = X_t - X_{t-s}$ which corresponds to the weights $a_0 = 1$ and $a_s = -1$ and $a_j = 0$ for all other j. Then the transfer function is clearly given by*

$$A(\lambda) = \sum_j a_j e^{-2\pi ij\lambda} = 1 - e^{-2\pi is\lambda} = 2i\sin(\pi s\lambda)e^{-\pi is\lambda},$$

where, for the last equality, the formula $1 - e^{i\theta} = -2i\sin(\theta/2)e^{i\theta/2}$ is used. Therefore the power transfer function equals

$$|A(\lambda)|^2 = 4\sin^2(\pi s\lambda) \qquad \text{for } -1/2 \le \lambda \le 1/2.$$

To understand this function, we only need to consider the interval $[0, 1/2]$ because it is symmetric on $[-1/2, 1/2]$.

When $s = 1$, the function $\lambda \mapsto |A(\lambda)|^2$ is increasing on $[0, 1/2]$. This means that first order differencing enhances the higher frequencies in the data and diminishes the lower frequencies. Therefore, it will make the data more wiggly.

For higher values of s, the function $A(\lambda)$ goes up and down and takes the value zero for $\lambda = 0, 1/s, 2/s, \ldots$. In other words, it eliminates all components of period s.

Example 2.2. *Now consider the moving average filter which corresponds to the coefficients $a_j = 1/(2q+1)$ for $|j| \le q$. The transfer function is*

$$\frac{1}{2q+1}\sum_{j=-q}^{q} e^{-2\pi ij\lambda} = \frac{S_{q+1}(\lambda) + S_{q+1}(-\lambda) - 1}{2q+1},$$

where it may be recalled (Lecture 19) that

$$S_n(g) := \sum_{t=0}^{n-1} \exp(2\pi igt) = \frac{\sin(\pi ng)}{\sin(\pi g)} e^{i\pi g(n-1)}$$

. Thus

$$S_n(g) + S_n(-g) = 2\frac{\sin(\pi ng)}{\sin(\pi g)}\cos(\pi g(n-1)),$$

which implies that the transfer function is given by

$$A(\lambda) = \frac{1}{2q+1}\left(2\frac{\sin(\pi(q+1)\lambda)}{\sin(\pi\lambda)}\cos(\pi q\lambda) - 1\right),$$

This function only depends on q and can be plotted for various values of q. For q large, it drops to zero very quickly. The interpretation is that the filter kills the high frequency components in the input process.

3 Spectral Densities of ARMA Processes

Suppose $\{X_t\}$ is a stationary ARMA process: $\phi(B)X_t = \theta(B)Z_t$ where the polynomials ϕ and θ have no common zeroes on the unit circle. Because of stationarity, the polynomial ϕ has no roots on the unit circle.

Let $U_t = \phi(B)X_t = \theta(B)Z_t$. Let us first write down the spectral density of $U_t = \phi(B)X_t$ in terms of that of $\{X_t\}$. Clearly, U_t can be viewed as the output of a filter applied to X_t. The filter is given by $a_0 = 1$ and $a_j = -\phi_j$ for $1 \leq j \leq p$ and $a_j = 0$ for all other j. Let $A_\phi(\lambda)$ denote the transfer function of this filter. Then we have

$$f_U(\lambda) = |A_\phi(\lambda)|^2 f_X(\lambda). \tag{4}$$

Similarly, using the fact that $U_t = \theta(B)Z_t$, we can write

$$f_U(\lambda) = |A_\theta(\lambda)|^2 f_Z(\lambda) = \sigma_Z^2 |A_\theta(\lambda)|^2 \tag{5}$$

where $A_\theta(\lambda)$ is the tranfer function of the filter with coefficients $a_0 = 1$ and $a_j = \theta_j$ for $1 \leq j \leq q$ and $a_j = 0$ for all other j. Equating (4) and (5), we obtain

$$f_X(\lambda) = \frac{|A_\theta(\lambda)|^2}{|A_\phi(\lambda)|^2}\sigma_Z^2 \qquad \text{for } -1/2 \leq \lambda \leq 1/2.$$

Now

$$A_\phi(\lambda) = 1 - \phi_1 e^{-2\pi i\lambda} - \phi_2 e^{-2\pi i(2\lambda)} - \cdots - \phi_p e^{-2\pi i(p\lambda)} = \phi(e^{-2\pi i\lambda}).$$

Similarly $A_\theta(\lambda) = \theta(e^{-2\pi i\lambda})$. As a result, we have

$$f_X(\lambda) = \sigma_Z^2 \frac{|\theta(e^{-2\pi i\lambda})|^2}{|\phi(e^{-2\pi i\lambda})|^2} \qquad \text{for } -1/2 \leq \lambda \leq 1/2.$$

Note that the denominator on the right hand side above is non-zero for all λ because of stationarity.

Example 3.1 (MA(1))**.** *For the MA(1) process: $X_t = Z_t + \theta Z_{t-1}$, we have $\phi(z) = 1$ and $\theta(z) = 1 + \theta z$. Therefore*

$$\begin{aligned} f_X(\lambda) &= \sigma_Z^2 \left|1 + \theta e^{2\pi i\lambda}\right|^2 \\ &= \sigma_Z^2 \left|1 + \theta\cos 2\pi\lambda + i\theta \sin 2\pi\lambda\right|^2 \\ &= \sigma_Z^2 \left[(1 + \theta\cos 2\pi\lambda)^2 + \theta^2 \sin^2 2\pi\lambda\right] \\ &= \sigma_Z^2 \left[1 + \theta^2 + 2\theta\cos 2\pi\lambda\right] \qquad \textit{for } -1/2 \leq \lambda \leq 1/2. \end{aligned}$$

Check that for $\theta = -1$, the quantity $1 + \theta^2 + 2\theta\cos(2\pi\lambda)$ equals the power transfer function of the first differencing filter.

Example 3.2 (AR(1))**.** *For AR(1): $X_t - \phi X_{t-1} = Z_t$, we have $\phi(z) = 1 - \phi z$ and $\theta(z) = 1$. Thus*

$$f_X(\lambda) = \sigma_Z^2 \frac{1}{|1 - \phi e^{2\pi i\lambda}|^2} = \frac{\sigma_Z^2}{1 + \phi^2 - 2\phi\cos 2\pi\lambda} \qquad \textit{for } -1/2 \leq \lambda \leq 1/2.$$

Example 3.3 (AR(2))**.** *For the AR(2) model: $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$, we have $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ and $\theta(z) = 1$. Here it can be shown that*

$$f_X(\lambda) = \frac{\sigma_Z^2}{1 + \phi_1^2 + \phi_2^2 - 2\phi_1(1 - \phi_2)\cos 2\pi\lambda - 2\phi_2 \cos 4\pi\lambda} \qquad \textit{for } -1/2 \leq \lambda \leq 1/2.$$