Spring 2013 Statistics 153 (Time Series) : Lecture Twenty Six

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We will continue discussion on the estimation of the spectral density. Estimators are given by:

$$\hat{f}(j/n) := \sum_{k=-m}^{m} W_m(k) I\left(\frac{j+k}{n}\right)$$

The set of weights $\{W_m(k)\}$ is often referred to as a kernel or a spectral window.

Simplest choice of $W_m(k)$ is

$$W_m(k) = \frac{1}{2m+1} \qquad \text{for } -m \le k \le m.$$

This window is called the *Daniell Spectral Window.*

One can get these estimates directly in R by using the function *spec.pgram* and *kernel.*

The bandwidth of a spectral window is defined as the standard deviation of the weighting distribution. It is actually this standard deviation that controls the bias of the estimator. This can be justified by a second order Taylor expansion as follows. The expected value of $\hat{f}(j/n)$ is

$$\mathbb{E}\hat{f}(j/n) = \sum_{k=-m}^{m} W_m(k) f\left(\frac{j+k}{n}\right)$$

Let $\lambda = j/n$ for ease of notation. Then by a second order Taylor expansion around λ, we get

$$\mathbb{E}\hat{f}(\lambda) = \sum_{k=-m}^{m} W_m(k) \left(f(\lambda) + \frac{k}{n} f'(\lambda) + \frac{k^2}{2n^2} f''(\lambda)\right)$$

If the weights are such that $\sum_k W_m(k) = 1$ and $\sum_k k W_m(k) = 0$ (satisfied for the Daniell kernel for example), then

$$\mathbb{E}\hat{f}(\lambda) - f(\lambda) = \frac{f''(\lambda)}{2} \sum_{k=-m}^{m} \left(\frac{k}{n}\right)^2 W_m(k)$$

The bandwidth of the kernel is given by

$$\sqrt{\sum_{k=-m}^{m} \left(\frac{k}{n}\right)^2 W_m(k)}.$$

For the Daniell kernel, the bandwidth is given by the standard deviation of the uniform distribution on $\{-m/n, -(m-1)/n, \dots, (m-1)/n, m/n\}$ which is very close to the standard deviation of the continuous uniform distribution on $[-m/n, m/n]$ which equals:

$$\sqrt{\frac{(2m)^2}{12n^2}} \approx \sqrt{\frac{L^2}{12n^2}} = \frac{L}{n\sqrt{12}}$$

Repeated use of the Daniell kernel yields non-uniform weights. For example, the Daniell kernel for $m = 1$ corresponds to the three weights $(1/3, 1/3, 1/3)$. Applying it to a sequence of numbers $\{u_t\}$ leads to the smoother:

$$\hat{u}_t = \frac{u_{t-1} + u_t + u_{t+1}}{3}.$$

Applying the Daniell kernel again to $\hat{u}_t$ gives

$$\hat{\hat{u}}_t := \frac{\hat{u}_{t-1} + \hat{u}_t + \hat{u}_{t+1}}{3} = \frac{1}{9}u_{t-2} + \frac{2}{9}u_{t-1} + \frac{3}{9}u_t + \frac{2}{9}u_{t+1} + \frac{1}{9}u_{t+2}.$$

Thus application of the Daniell kernel is equivalent to applying the kernel $(1/9, 2/9, 3/9, 2/9, 1/9)$ to the data. This is a non-uniform kernel with a higher bandwidth. Note also that these weights equal the convolution of the Daniell kernel. In other words, if X_1 and X_2 both have the pmfs $(1/3, 1/3, 1/3)$, then $X_1 + X_2$ has the pmf $(1/9, 2/9, 3/9, 2/9, 1/9)$. If we keep applying the Daniell kernel repeatedly, we get spectral windows that look very much like a gaussian pdf.

Another common kernel choice is the modified Daniell kernel which puts half-weights at the end-points. The book also talks about the Dirichlet kernel and the Fejer kernel.