Spring 2013 Statistics 153 (Time Series) : Lecture Twenty One

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1 The Periodogram

In the last class, we saw the following connection between the DFT and the sample autocovariance function:

$$\frac{|b_j|^2}{n} = \sum_{h:|h|<n} \hat{\gamma}(h) \exp\left(-\frac{2\pi i j h}{n}\right) \qquad \text{for } j = 1, \dots, [n/2].$$

The function

$$I(j/n) := \frac{|b_j|^2}{n} = \sum_{h:|h|<n} \hat{\gamma}(h) \exp\left(-\frac{2\pi i j h}{n}\right) \qquad \text{for } j = 1, \dots, [n/2] \tag{1}$$

is called the *periodogram* of the data $x_0, x_1, \dots, x_{n-1}$. **The periodogram gives the strengths of sinusoids at various frequencies in the data**.

2 The Spectral Density

Suppose $\{X_t\}$ is a doubly infinite sequence of random variables that is stationary. Let $\{\gamma(h)\}$ denote their autocovariance function. In analogy with the definition (1) of the Periodogram, we define

$$f(\lambda) := \sum_{h=-\infty}^{\infty} \gamma(h) \exp\left(-2\pi i \lambda h\right) \qquad \text{for } -1/2 \leq \lambda \leq 1/2 \tag{2}$$

and call this quantity the *Spectral Density* of the stationary sequence of random variables, $\{X_t\}$. Because the complex exponentials $e^{-2\pi i \lambda h}$ are all periodic in λ with period 1, we only need to define f on an interval of length 1 and, by convention, we focus on the interval $[-1/2, 1/2]$. In fact, note that f is symmetric and we really only need to worry about $[0, 1/2]$.

In analogy with the periodogram, the spectral density will give the strengths of sinusoids at various frequencies in the data.

We have defined the spectral density in terms of the autocovariance function. It turns that the autocovariance function can also be obtained from the spectral density: To see this, just multiply both sides of (2) by $e^{2\pi i \lambda k}$ for a fixed k and integrate from $\lambda = -1/2$ to $\lambda = 1/2$ to get:

$$\gamma(k) = \int_{1/2}^{1/2} e^{2\pi i \lambda k} f(\lambda) d\lambda \tag{3}$$

In other words, the autocovariance function and the spectral density provide equivalent information about the stationary process $\{X_t\}$.

There is one problem however with the definition of the spectral density. The infinite sum in (2) need not always make sense. Indeed, the complex exponentials $\exp(-2\pi i\lambda h)$ always have a magnitude of 1 and so the sum (2) only makes sense when $\{\gamma(h)\}$ decay sufficiently quickly. A sufficient (but not necessary) condition for (2) to make sense is $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$.

It is not too hard to find examples where the sum on the right hand side in (2) does not make sense. For example, consider the process where $X_t = A\cos 2\pi\lambda_1 t + B\sin 2\pi\lambda_1 t$ where A and B are uncorrelated random variables both with mean 0 and variance σ^2 and $0 < \lambda_1 < 1/2$ is a fixed (non-random) frequency. This process is clearly stationary and its autocovariance function equals $\gamma(h) = \sigma^2 \cos 2\pi\lambda_1 h$. Clearly, this does not decay fast enough and $\sum_h \gamma(h)\exp(-2\pi i\lambda h)$ does not make sense for any λ.

It turns out that one may not be able to define a spectral density for every stationary process $\{X_t\}$. But one can always define a Spectral Distribution Function. The analogy is to random variables (Not all random variables have densities but they all have distribution functions).

Before defining the spectral distribution function, let us briefly discuss elementary expectations.

2.1 Review of Expectations

Let X be a random variable. The distribution function of X is defined as $F(x) = \mathbb{P}\{X \le x\}$. The function F is non-negative, right-continuous, non-decreasing and satisfies:

$$\lim_{x\to-\infty} F(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} F(x) = 1.$$

The expectation of a function $g(X)$ of X is sometimes denoted by:

$$\mathbb{E}g(X) = \int g(x)dF(x).$$

The computation of this expectation is rather easy in the following two cases:

1. X is a discrete random variable taking values $x_1 < \cdots < x_k$ with probabilities $p_1, \dots, p_k$. In this case, F has a jump of size p_i at x_i and is constant between x_i and x_{i+1}. And, $\int g(x)dF(x) = \sum_i g(x_i)p_i$.
2. X has a density f. In this case, $F(x) = \int_{-\infty}^{x} f(x)dx$ and $\mathbb{E}g(X) = \int g(x)f(x)dx$.

The quantity $\int g(x)dF(x)$ can also be defined for F which are non-negative, right-continuous, non-decreasing and satisfy:

$$\lim_{x\to-\infty} F(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} F(x) = \boldsymbol{\sigma^2}$$

for some $\sigma^2 > 0$. In this case F/σ^2 is a distribution function and thus the integral $\int g(x)dF(x)$ can be defined as

$$\int g(x)dF(x) = \sigma^2 \int g(x)d\tilde{F}(x) \qquad \text{where } \tilde{F}(x) = \frac{F(x)}{\sigma^2}.$$

2.2 Spectral Distribution Function and Spectral Density

Let $\{X_t\}$ be a stationary sequence of random variables and let $\gamma_X(h) = \mathrm{cov}(X_t, X_{t+h})$ denote the autocovariance function.

A theorem due to Herglotz (sometimes attributed to Bochner) states that **every** autocovariance function γ_X can be written as:

$$\gamma_X(h) = \int_{-1/2}^{1/2} e^{2\pi i h\lambda} dF(\lambda),$$

where $F(\cdot)$ is a non-negative, right-continuous, non-decreasing function on $[-1/2, 1/2]$ with $F(-1/2) = 0$ and $F(1/2) = \gamma_X(0)$. Moreover, F is uniquely determined by γ_X.

If F has a density f, then f is called the *Spectral Density* of $\{X_t\}$.

A sufficient condition (but not necessary) for the existence of the spectral density is the condition $\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty$. And in this case, the spectral density exists and is given by the formula (2).

2.3 Discrete Spectrum Example

Suppose

$$X_t = \sum_{j=1}^{m} \left(A_j \cos(2\pi\lambda_j t) + B_j \sin(2\pi\lambda_j t)\right) \qquad \text{for } t = \ldots, -2, -1, 0, 1, 2, \ldots,$$

where the frequencies $0 < \lambda_1 < \cdots < \lambda_m < 1/2$ are fixed and $A_1, B_1, A_2, B_2, \ldots, A_m, B_m$ are *uncorrelated* random variables with common mean 0 and $\text{var}(A_j) = \sigma_j^2 = \text{var}(B_j)$. The covariance between X_t and X_{t+h} equals:

$$\sum_j \sigma_j^2 \left(\cos(2\pi\lambda_j t)\cos(2\pi\lambda_j(t+h)) + \sin(2\pi\lambda_j t)\sin(2\pi\lambda_j(t+h))\right) = \sum_j \sigma_j^2 \cos(2\pi\lambda_j h).$$

Because this covariance does not depend on t, the process $\{X_t\}$ is stationary with autocovariance function $\gamma_X(h) = \sum_{j=1}^{m} \sigma_j^2 \cos(2\pi\lambda_j h)$. This autocovariance function $\gamma_X(h)$ can be written as:

$$\gamma_X(h) = \sum_{j=1}^{m} \sigma_j^2 \left(\frac{e^{2\pi i\lambda_j h} + e^{-2\pi i\lambda_j h}}{2}\right)$$

Thus $\gamma_X(h)$ equals $\int_{-1/2}^{1/2} e^{2\pi i h\lambda} dF(\lambda)$ where F corresponds to the discrete distribution which takes values

$$-\lambda_m < \cdots < -\lambda_1 < \lambda_1 < \cdots < \lambda_m$$

with weights

$$\frac{\sigma_m^2}{2}, \ldots, \frac{\sigma_1^2}{2}, \frac{\sigma_1^2}{2}, \ldots, \frac{\sigma_m^2}{2}.$$

Note that this is a symmetric distribution. Thus the spectral distribution function puts mass only at the frequencies that are present in $\{X_t\}$. Moreover, the mass at a particular frequency λ_j is proportional to the variance σ_j^2 at that frequency. The total mass of the spectral distribution is:

$$\frac{\sigma_m^2}{2} + \cdots + \frac{\sigma_1^2}{2} + \frac{\sigma_1^2}{2} + \cdots + \frac{\sigma_m^2}{2} = \sigma_1^2 + \cdots + \sigma_m^2 = \gamma_X(0).$$

2.4 White Noise

For white noise $\gamma_X(h) = 0$ for $h \neq 0$ and $\gamma_X(0) = \sigma^2$. Thus $\sum_h |\gamma_X(h)| < \infty$ and the spectral density is given by:

$$f(\lambda) = \sum_{h=-\infty}^{\infty} \gamma_X(h) \exp(-2\pi i\lambda h) = \gamma_X(0) = \sigma^2 \qquad \text{for all } -1/2 \leq \lambda \leq 1/2.$$

The idea is that all frequencies are present in white noise in equal amounts.

2.5 Spectral Density for ARMA processes

Theorem 2.1. *Let $\{Y_t\}$ be a mean-zero, stationary process with Spectral Distribution Function F_Y. Define $\{X_t\}$ by*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} \qquad \text{where } \sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

Then $\{X_t\}$ is stationary with Spectral Distribution Function:

$$F_X(\lambda) = \int_{-1/2}^{\lambda} \left| \sum_j \psi_j e^{2\pi i j \lambda} \right|^2 dF_Y(\lambda). \qquad \text{for } -1/2 \leq \lambda \leq 1/2.$$

Proof. The autocovariance of X_t is (note that X_t has mean zero because $\{Y_t\}$ was assumed to have zero mean):

$$\gamma_X(h) = \mathbb{E} X_t X_{t+h} = \mathbb{E}(\sum_j \psi_j Y_{t-j})(\sum_k \psi_k Y_{t+h-k}) = \sum_{j,k} \psi_j \psi_k \gamma_Y(h-k+j).$$

By the definition of the spectral distribution function, we can write:

$$\gamma_Y(h-k+j) = \int_{-1/2}^{1/2} e^{2\pi i(h-k+j)\lambda} dF_Y(\lambda).$$

Therefore,

$$\begin{aligned}
\gamma_X(h) &= \sum_{j,k} \psi_j \psi_k \int_{-1/2}^{1/2} e^{2\pi i(h-k+j)\lambda} dF_Y(\lambda) \\
&= \int_{-1/2}^{1/2} e^{2\pi i h \lambda} \sum_{j,k} \psi_j \psi_k e^{-2\pi i k \lambda} e^{2\pi i j \lambda} dF_Y(\lambda) \\
&= \int_{-1/2}^{1/2} e^{2\pi i h \lambda} \left(\sum_j \psi_j e^{2\pi i j \lambda} \right) \left(\sum_k \psi_k e^{-2\pi i k \lambda} \right) dF_Y(\lambda) \\
&= \int_{-1/2}^{1/2} e^{2\pi i h \lambda} \left| \sum_j \psi_j e^{2\pi i j \lambda} \right|^2 dF_Y(\lambda)
\end{aligned}$$

This is of the form:

$$\gamma_X(h) = \int_{-1/2}^{1/2} e^{2\pi i h \lambda} dF_X(\lambda)$$

with

$$dF_X(\lambda) = \left| \sum_j \psi_j e^{2\pi i j \lambda} \right|^2 dF_Y(\lambda).$$

The proof is complete. $\square$

It follows from the above theorem that if $\{Y_t\}$ has a spectral density f_Y, then X_t also has a spectral density that is given by

$$f_X(\lambda) = \left| \sum_{j=-\infty}^{\infty} \psi_j e^{2\pi i j \lambda} \right|^2 f_Y(\lambda) \qquad \text{for } -1/2 \leq \lambda \leq 1/2.$$

If we use the notation $\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j$, then the spectral density $f_X(\lambda)$ can be written as:

$$f_X(\lambda) = |\psi(e^{2\pi i\lambda})|^2 f_Y(\lambda)$$