Spring 2013 Statistics 153 (Time Series) : Lecture Twenty Four

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In the last class, we discussed the problem of nonparametrically estimating a spectral density. The natural estimator is:

$$I(\lambda) := \sum_{h:|h|<n} \hat{\gamma}(h) e^{-2\pi i \lambda h} \qquad \text{for } -1/2 \leq \lambda \leq 1/2$$

which for $\lambda = j/n \in (0, 1/2]$ coincides with the periodogram:

$$I(j/n) = \frac{|b_j|^2}{n} \qquad \text{where } b_j = \sum_t x_t \exp\left(-\frac{2\pi i j t}{n}\right)$$

The key result about the periodogram is that under some regularity conditions which hold for all ARMA processes under the gaussian noise, it can be shown that when n is large, the random variables:

$$\frac{2I(j/n)}{f(j/n)} \qquad \text{for } 0 < j < n/2$$

are approximately independently distributed according to the χ^2_2 distribution. As a result, $I(\lambda)$ is not a good estimator of $f(\lambda)$.

We studied two modifications of the periodogram:

1. **Moving average smoothing**: Choose an integer $m \geq 1$ and estimate $f(j/n)$ by

$$\hat{f}(j/n) := \frac{1}{2m+1} \sum_{k=-m}^{m} I\left(\frac{j+k}{n}\right)$$

or more generally

$$\hat{f}(j/n) := \sum_{k=-m}^{m} W(k) I\left(\frac{j+k}{n}\right)$$

where $W(k)$ are nonnegative weights summing to one. This estimator is based on the approximate representation $I(j/n) \approx f(j/n) + U_j f(j/n)$ for $0 < j < n/2$ where $\{U_j\}$ is white noise.

2. **Lag Window Spectral Density Estimator**: Choose an integer $r \geq 1$ and estimate $f(j/n)$ by

$$\hat{f}(j/n) := \sum_{h:|h|\leq r} \hat{\gamma}(h) \exp\left(-2\pi i \lambda h\right)$$

or more generally

$$\hat{f}(j/n) := \sum_{h:|h|\leq r} w\left(\frac{h}{r}\right) \hat{\gamma}(h) \exp\left(-2\pi i \lambda h\right)$$

where $w(x)$ is a symmetric i.e., $w(x) = w(-x)$ function satisfying $w(0) = 1$, $|w(x)| \leq 1$ and $w(x) = 0$ for $|x| > 1$. This estimator is based on the formula:

$$f(\lambda) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \lambda h}$$

and the idea that $\gamma(h)$ for large h become close to zero (because $\sum_h |\gamma(h)| < \infty$) and they are also difficult to estimate from the data.

1 Equivalence of these Two Estimators

We shall now show that these two ways of improving the periodogram: by smoothing it and the lag window spectral density estimator are essentially the same. To see this, we first need an *inverse* relationship between $I(\lambda)$ and $\hat{\gamma}(h)$. We have defined $I(\lambda)$ as

$$I(\lambda) := \sum_{h:|h|<n} \hat{\gamma}(h) e^{-2\pi i \lambda h} \qquad \text{for } -1/2 \leq \lambda \leq 1/2.$$

It is possible to invert this formula to write $\hat{\gamma}(k)$ in terms of $I(\lambda)$. Fix an integer k with $|k| < n$ and multiply both sides of the above formula by $e^{2\pi i \lambda k}$. Integrating the resulting expression with respect to λ from $-1/2$ to $1/2$, we get

$$\int_{-1/2}^{1/2} e^{2\pi i \lambda k} I(\lambda) d\lambda = \sum_{h:|h|<n} \hat{\gamma}(h) \int_{-1/2}^{1/2} e^{2\pi i \lambda (k-h)} d\lambda = \hat{\gamma}(k).$$

This therefore implies

$$\hat{\gamma}(k) = \int_{-1/2}^{1/2} e^{2\pi i \lambda k} I(\lambda) d\lambda. \tag{1}$$

In other words, the function $I(\lambda)$ is the spectral density corresponding to the sample autocorrelation function. Using the formula (1), we can write the lag window spectral density estimator as

$$\begin{aligned}
\tilde{f}(\lambda) &= \sum_{h:|h|\leq r} w\left(\frac{h}{r}\right) \hat{\gamma}(h) e^{-2\pi i \lambda h} \\
&= \sum_{h:|h|\leq r} w\left(\frac{h}{r}\right) \int_{-1/2}^{1/2} e^{2\pi i \rho h} I(\rho) d\rho \; e^{-2\pi i \lambda h} \\
&= \int_{-1/2}^{1/2} I(\rho) \sum_{h:|h|\leq r} w\left(\frac{h}{r}\right) e^{2\pi i (\rho - \lambda) h} d\rho.
\end{aligned}$$

By the change of variable $\rho = \lambda + u$, we get

$$\tilde{f}(\lambda) = \int I(\lambda + u) \sum_{h:|h|\leq r} w\left(\frac{h}{r}\right) e^{2\pi i u h} du.$$

Letting

$$W(u) = \sum_{h:|h|\leq r} w\left(\frac{h}{r}\right) e^{2\pi i u h},$$

we get that

$$\tilde{f}(\lambda) = \int I(\lambda + u) W(u) du.$$

Thus the lag window spectral density estimator $\tilde{f}$ can also be thought of as obtained by smoothing the periodogram.

2 Approximate Confidence Intervals for $f(j/n)$

Recall that the random variables

$$\frac{2I(j/n)}{f(j/n)} \qquad \text{for } 0 < j < n/2$$

are approximately independently distributed according to the χ^2_2 distribution.

Therefore, approximately

$$\hat{f}(j/n) = \frac{1}{2m+1} \sum_{k=-m}^{m} I\left(\frac{j+k}{n}\right) \approx \frac{f(j/n)}{2(2m+1)} \sum_{k=-m}^{m} \frac{2I((j+k)/n)}{f((j+k)/n)}.$$

This would allow us to approximate the distribution of $\hat{f}(j/n)$ in the following way:

$$2(2m+1)\frac{\hat{f}(j/n)}{f(j/n)} \sim \chi^2_{2(2m+1)}.$$

If $\chi^2_{2(2m+1)}(\alpha/2)$ and $\chi^2_{2(2m+1)}(1-\alpha/2)$ satisfy

$$\mathbb{P}\left\{\chi^2_{2(2m+1)}(\alpha/2) \le \chi^2_{2(2m+1)} \le \chi^2_{2(2m+1)}(1-\alpha/2)\right\} = 1-\alpha,$$

then we conclude that approximately

$$\mathbb{P}\left\{\chi^2_{2(2m+1)}(\alpha/2) \le 2(2m+1)\frac{\hat{f}(j/n)}{f(j/n)} \le \chi^2_{2(2m+1)}(1-\alpha/2)\right\} \approx 1-\alpha.$$

This would lead to the following confidence interval for $f(j/n)$ of level approximately $1-\alpha$:

$$2(2m+1)\frac{\hat{f}(j/n)}{\chi^2_{2(2m+1)}(1-\alpha/2)} \le f(j/n) \le 2(2m+1)\frac{\hat{f}(j/n)}{\chi^2_{2(2m+1)}(\alpha/2)}.$$