Spring 2013 Statistics 153 (Time Series) : Lecture Twenty Five

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Suppose $x_1, \ldots, x_n$ are data that we assume come from a stationary process $\{X_t\}$ with mean zero and variance σ^2.

Previously, we studied the stationary ARMA models for modelliing such data. Today we ask if there are any other natural ways of modelling such data. In particular, we investigate if it makes sense to use sines and cosines to model $x_1, \ldots, x_n$.

The simplest stationary model using sines and cosines is

$$X_t = A \cos(2\pi \lambda t) + B \sin(2\pi \lambda t)$$

where $0 \leq \lambda \leq 1/2$ is a fixed constant and A and B are uncorrelated random variables with mean 0 and variance σ^2. We have seen many times in the past that $\{X_t\}$ is stationary with mean 0 and variance σ^2.

More complicated stationary models with sines and cosines can be constructed by taking linear combinations of the form

$$X_t = \sum_{j=1}^{m} \left(A_j \cos(2\pi \lambda_j t) + B_j \sin(2\pi \lambda_j t) \right) \tag{1}$$

where $0 \leq \lambda \leq 1/2$ is a fixed constant and $A_1, B_1, A_2, B_2, \ldots, A_m, B_m$ are all uncorrelated random variables with mean zero and

$$\text{var}(A_j) = \text{var}(B_j) = \sigma_j^2.$$

Let $\sum_{j=1}^{m} \sigma_j^2 = \sigma^2$ so that the variance of the process $\{X_t\}$ equals σ^2.

It turns out that the model (1) can approximate any stationary model provided m is large enough and $\lambda_1, \ldots, \lambda_m$ and $\sigma_1^2, \ldots, \sigma_m^2$ are chosen appropriately. For example, the choices

$$\lambda_j = \frac{j}{2m} \text{ and } \sigma_j^2 = \frac{\sigma^2}{m} \qquad \text{for } j = 1, \ldots, m$$

for m large lead to a very good approximation of the white noise model.

To approximate a general stationary process using sines and cosines, we can use its spectral density. We know that the spectral density $f(\lambda)$ satisfies

$$\int_{-1/2}^{1/2} f(\lambda) d\lambda = \sigma^2.$$

For real valued stationary processes, the spectral density is symmetric around zero. Therefore,

$$\int_{0}^{1/2} f(\lambda) d\lambda = \frac{\sigma^2}{2}.$$

Approximating the integral on the left hand side above by a Riemann sum (assuming such an approximation is valid), we obtain

$$\frac{\sigma^2}{2} = \sum_{j=1}^{m} \int_{\lambda_{j-1}}^{\lambda_j} f(\lambda) d\lambda \approx \frac{1}{2m} \sum_{j=1}^{m} f(\lambda_j) \qquad \text{where } \lambda_j = \frac{j}{2m}.$$

If we now take

$$\lambda_j = \frac{j}{2m} \text{ and } \sigma_j^2 = \frac{f(\lambda_j)}{m}$$

Then, for large m, the process (1) will approximate the stationary process with spectral density f. This method can be used to, for example, simulate ARMA processes without using the *arima.sim* function in R.

When m equals ∞, the process (1) can be defined to make sense so that it has exactly the same spectral density f. But this requires stochastic integration and is beyond the scope of this class.

The above analysis conveys the key role of the spectral density for the study of stationary processes. It essentially tells us all that there is to know about the stationary process.