Spring 2013 Statistics 153 (Time Series) : Lecture Twenty

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1 DFT Again

Data is denoted by $x_0, x_1, \ldots, x_{n-1}$.

DFT is denoted by $b_0, b_1, \ldots, b_{n-1}$.

The DFT is calculated from data by

$$b_j := \sum_{t=0}^{n-1} x_t \exp\left(-\frac{2\pi i j t}{n}\right) \qquad \text{for } j = 0, 1, \ldots, n-1. \tag{1}$$

The data is calculated from the DFT by

$$x_t = \frac{1}{n} \sum_{j=0}^{n-1} b_j \exp\left(\frac{2\pi i j t}{n}\right) \qquad \text{for } t = 0, 1, \ldots, n-1.$$

Remember that $b_0 = x_0 + \cdots + x_{n-1}$ and $b_{n-j} = \bar{b}_j$ for $1 \leq j \leq n-1$. In the textbook, the formula for the DFT is given by (1) with an extra factor of $n^{-1/2}$. I have dropped this factor to make the definition compatible with the R function *fft*.

For odd values of n, the DFT is comprised of the real number b_0 and the $(n-1)/2$ complex numbers $b_1, \ldots, b_{(n-1)/2}$.

For even values of n, the DFT consists of two real numbers b_0 and $b_{n/2}$ and the $(n-2)/2$ complex numbers $b_1, \ldots, b_{(n-2)/2}$.

2 What does the DFT do?

Suppose $x_t = R\cos(2\pi f_0 t + \Phi)$ for $t = 0, 1, \ldots, n-1$. We have seen in the last class that we only have to consider frequencies in the range $0 \leq f_0 \leq 1/2$ (because every other frequency has an *alias* in the interval $[0, 1/2]$).

Assume first that f_0 is of the form k/n for some k where $0 \leq k/n \leq 1/2$. Then the DFT is given by

$$\begin{aligned} b_j &= \sum_{t=0}^{n-1} R\cos(2\pi(k/n)t + \Phi) \exp\left(-2\pi i (j/n) t\right) \\ &= \frac{Re^{i\Phi}}{2} \sum_{t=0}^{n-1} \exp\left(2\pi i t \frac{j-k}{n}\right) + \frac{Re^{-i\Phi}}{2} \sum_{t=0}^{n-1} \exp\left(-2\pi i t \frac{j+k}{n}\right). \end{aligned}$$

Note that we do not need to consider the DFT b_j for $j/n > 1/2$. So we assume that $0 \leq j/n \leq 1/2$. Because the original cosine wave was assumed to have frequency in the range $[0, 1/2]$, we have $0 \leq k/n \leq 1/2$. Check that $0 < (k+j)/n < 1$ when $j \neq k$. Because of all this, we get that the second term above is always zero and the first term equals zero when $j \neq k$ and equals $Re^{i\Phi}n/2$ when $j = k$. Therefore the DFT of the cosine wave with a frequency k/n for $0 \leq k/n \leq 1/2$ is $b_k = nRe^{i\Phi}/2$ for and $b_j = 0$ for $j \neq k$ and $0 \leq j/n \leq 1/2$.

Now consider data that is linear combination of multiple frequencies:

$$x_t = \sum_{l=1}^{m} R_l \cos(2\pi t(k_l/n) + \Phi_l) \tag{2}$$

where each k_l is an integer satisfying $0 \leq k_l/n \leq 1/2$. Because the definition of the DFT is linear in the data $\{x_t\}$, it follows that the DFT of (2) is given by

$$b_j = \begin{cases} nR_l e^{i\Phi_l}/2 & \text{if } j = k_l \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq j/n \leq 1/2$.

This shows that the DFT picks out the frequencies present in the data. The strength (absolute value) of the DFT at a frequency is proportional to the amplitude (R_l) of the cosine wave at that frequency.

3 Interpreting the DFT

The DFT writes the given data in terms of sinusoids with frequencies of the form k/n. Frequencies of the form k/n are called Fourier frequencies.

Suppose that we are given a dataset $x_0, \dots, x_{n-1}$. We have calculated its DFT: $b_0, b_1, \dots, b_{n-1}$ and we have plotted $|b_j|$ for $j = 1, \dots, (n-1)/2$ for odd n and for $j = 1, \dots, n/2$ for even n.

If we see a single spike in this plot, say at b_k, we are sure that the data is a sinusoid with frequency k/n.

If we get two spikes, say at b_{k_1} and b_{k_2}, then the data is slightly more complicated: it is a linear combination of two sinusoids at frequencies k_1/n and k_2/n with the strengths of these sinusoids depending on the size of the spikes.

Multiple spikes indicate that the data is made up of many sinusoids at Fourier frequencies and, in general, this means that the data is more complicated.

However, sometimes one can see multiple spikes in the DFT even when the structure of the data is not very complicated. A typical example is leakage due to the presence of a sinusoid at a non-Fourier frequency.

The DFT of a sinusoid at a non-Fourier frequency is calculated in the following way: Consider the signal $x_t = e^{2\pi f_0 t}$ where $f_0 \in [0, 1/2]$ is not necessarily of the form k/n for any k. Its DFT is given by

$$b_j := \sum_{t=0}^{n-1} x_t e^{-2\pi i t(j/n)} = \sum_{t=0}^{n-1} e^{2\pi i(f_0 - (j/n))t}.$$

If we denote the function

$$S_n(g) := \sum_{t=0}^{n-1} e^{2\pi i g t} \tag{3}$$

then we can write

$$b_j = S_n(f_0 - (j/n)).$$

The function $S_n(g)$ can clearly be evaluated using the geometric series formula to be

$$S_n(g) = \frac{e^{2\pi i g n} - 1}{e^{2\pi i g} - 1}$$

Because

$$e^{i\theta} - 1 = \cos\theta + i\sin\theta - 1 = 2e^{i\theta/2}\sin\theta/2,$$

we get

$$S_n(g) = \frac{\sin \pi n g}{\sin \pi g} e^{i\pi g(n-1)}$$

Thus the absolute value of the DFT of $y_t = e^{2\pi i f_0 t}$ is given by

$$|b_j| = |S_n(f_0 - (j/n))| = \left| \frac{\sin \pi n(f_0 - (j/n))}{\sin \pi(f_0 - (j/n)))} \right|$$

This expression becomes meaningless when $f_0 = j/n$. But when $f_0 = f$, the value of $S_n(f_0 - j/n)$ can be directly be calculated from (3) to be equal to n.

The behavior of this DFT can be best understood by plotting the function $g \mapsto (\sin \pi n g)/(\sin \pi g)$.

4 Leakage Reduction by Hanning

Hanning is a technique to reduce leakage which says: Multiply the data by the *window* or *fader*:

$$w_t = 1 - \cos(2\pi t/n) \qquad \text{for } t = 0, 1, \ldots, n-1$$

and then take the DFT.

Why does it work? The following is the DFT of $y_t = w_t e^{2\pi i f_0 t}$ (below f stands for j/n)

$$\begin{aligned}
b_y(f) &:= \sum_{t=0}^{n-1} w_t e^{2\pi i f_0 t} e^{-2\pi i f t} \\
&= \sum_{t=0}^{n-1} (1 - \cos(2\pi t/n))\, e^{2\pi i (f_0 - f)t} \\
&= \sum_{t=0}^{n-1} e^{2\pi i (f_0 - f)t} - \frac{1}{2}\sum_{t=0}^{n-1} e^{2\pi i (f_0 - f + 1/n)t} - \frac{1}{2}\sum_{t=0}^{n-1} e^{2\pi i (f_0 - f - 1/n)t} \\
&= S_n(f_0 - f) - \frac{1}{2} S_n(f_0 - f + 1/n) - \frac{1}{2} S_n(f_0 - f - 1/n).
\end{aligned}$$

Clearly $b_y(f_0) = n = b(f_0)$ because $S_n(1/n) = 0$. Suppose $g = f_0 - f$. To eliminate leakage, we need to make sure that $b_y(f)$ is close to zero when f is not equal to f_0. This is not quite possible but what we shall heuristically show is that when $|f - f_0|$ is reasonably large compared to $1/n$, then $b_f(f)$ is close to zero.

Let g denote $f - f_0$ so that

$$b_y(f) = S_n(g) - \frac{1}{2} S_n(g - 1/n) - \frac{1}{2} S_n(g + 1/n). \tag{4}$$

We derived in the last section that

$$S_n(g) = \frac{\sin \pi n g}{\sin \pi g} e^{\pi i g(n-1)}.$$

Therefore,

$$S_n(g-1/n) = \frac{\sin \pi n(g-1/n)}{\sin \pi (g-1/n)} e^{\pi i(g-1/n)(n-1)}$$

Now $\sin \pi n(g-1/n) = -\sin \pi n g$ and if $1/n$ is small compared to g, then $\sin \pi(g-1/n) \approx \sin \pi g$. Also when $1/n$ is small compared to g, we have

$$e^{\pi i(g-1/n)(n-1)} = e^{\pi i g(n-1)} e^{-i\pi(n-1)/n} \approx e^{\pi i g(n-1)} e^{-i\pi} = -e^{\pi i g(n-1)}.$$

Therefore, if $1/n$ is small compared to g, we have

$$S_n(g-1/n) \approx S_n(g)$$

and similarly $S_n(g+1/n) \approx S_n(g)$. Therefore, from (4), we get $b_y(f) \approx 0$ provided $f - f_0$ is not too small compared to $1/n$. Also $b_y(f_0) = b(f_0)$. Thus leakage is reduced.

It is usually the case however that for f close to f_0, $b_y(f)$ is much larger than $b(f)$. Thus the price that is paid for the reduction in leakage is that the peaks are slightly rounder at the top compared to the peaks without hanning.

5 DFT and Sample Autocovariance Function

We show below that

$$\frac{|b_j|^2}{n} = \sum_{|h|<n} \hat{\gamma}(h) \exp\left(-\frac{2\pi i j h}{n}\right) \qquad \text{for } j = 1, \dots, n-1$$

where $\hat{\gamma}(h)$ is the sample autocovariance function. This gives an important connection between the dft and the sample autocovariance function.

To see this, observe first, by the formula for the sum of a geometric series, that

$$\sum_{t=0}^{n-1} \exp\left(-\frac{2\pi i j t}{n}\right) = 0 \qquad \text{for } j = 1, \dots, n-1.$$

In other words, if the data is constant i.e., $x_0 = \cdots = x_{n-1}$, then b_0 equals $n x_0$ and b_j equals 0 for all other j. Because of this, we can write:

$$b_j = \sum_{t=0}^{n-1} (x_t - \bar{x}) \exp\left(-\frac{2\pi i j t}{n}\right) \qquad \text{for } j = 1, \dots, n-1.$$

Therefore, for $j = 1, \dots, n-1$, we write

$$\begin{aligned}
|b_j|^2 = b_j \bar{b}_j &= \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} (x_t - \bar{x})(x_s - \bar{x}) \exp\left(-\frac{2\pi i j t}{n}\right) \exp\left(\frac{2\pi i j s}{n}\right) \\
&= \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} (x_t - \bar{x})(x_s - \bar{x}) \exp\left(-\frac{2\pi i j (t-s)}{n}\right) \\
&= \sum_{h=-(n-1)}^{n-1} \sum_{t,s: t-s=h} (x_t - \bar{x})(x_{t-h} - \bar{x}) \exp\left(-\frac{2\pi i j h}{n}\right) \\
&= n \sum_{|h|<n} \hat{\gamma}(h) \exp\left(-\frac{2\pi i j h}{n}\right).
\end{aligned}$$