Spring 2013 Statistics 153 (Time Series) : Lecture Fourteen

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1 Asymptotic Distribution of ARMA ML Estimates

Let $\beta = (\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q)$. The distribution of $\hat{\beta}$ is approximately normal with mean β and variance covariance matrix $\sigma_Z^2 \Gamma_{p,q}^{-1}/n$ where $\Gamma_{p,q}$ is a $(p+q) \times (p+q)$ matrix of the form:

$$\begin{pmatrix} \Gamma_{\phi\phi} & \Gamma_{\phi\theta} \\ \Gamma_{\theta\phi} & \Gamma_{\theta\theta} \end{pmatrix}$$

The (i,j)th entry of the $p \times p$ matrix $\Gamma_{\phi\phi}$ equals $\gamma_A(i-j)$ where $\{A_t\}$ is the AR(p) process: $\phi(B)A_t = Z_t$. Similarly, the $q \times q$ matrix $\Gamma_{\theta\theta}$ has (i,j)th entry equalling $\gamma_B(i-j)$ for the AR(q) process $\theta(B)B_t = Z_t$. The (i,j)th entry of $\Gamma_{\phi\theta}$ equals the covariance between A_i and B_j.

1.1 Special Cases

The result in particular states that the variance-covariance matrix for AR and MA models will be very similar (the only difference is in signs). In the MA(1) case:

$$\Gamma_\theta = \Gamma_1 = \sigma_Z^2/(1-\theta^2).$$

Thus $\hat{\theta}$ is approximately normal with mean θ and variance $(1-\theta^2)/n$.

For MA(2), $(\hat{\theta}_1, \hat{\theta}_2)$ is approximately normal with mean (θ_1, θ_2) and variance-covariance matrix is $1/n$ times

$$\begin{pmatrix} 1-\theta_2^2 & \theta_1(1-\theta_2) \\ \theta_1(1-\theta_2) & 1-\theta_2^2 \end{pmatrix}$$

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For ARMA(1, 1), to calculate $\Gamma_{\phi\theta}$, we must find the covariance between A_1 and B_1 where $A_1 - \phi A_0 = Z_t$ and $B_1 + \theta B_0 = Z_t$. Write

$$\Gamma_{\phi\theta} = \text{cov}(A_1, B_1) = \text{cov}(\phi A_0 + Z_1, -\theta B_0 + Z_t) = -\phi\theta\Gamma_{\phi\theta} + \sigma_Z^2$$

which gives $\Gamma_{\phi\theta} = \sigma_Z^2/(1+\phi\theta)$. This gives that $(\hat{\phi}, \hat{\theta})$ is approximately normal with mean (ϕ, θ) and variance-covariance matrix is $1/n$ times the **inverse** of

$$\begin{pmatrix} (1-\phi^2)^{-1} & (1+\phi\theta)^{-1} \\ (1+\phi\theta)^{-1} & (1-\theta^2)^{-1} \end{pmatrix}$$

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2 ARIMA Models

ARIMA is essentially differencing plus ARMA. We have seen previously that differencing is commonly used on time series data to remove trends and seasonality.

For example, differencing can be used for

1. Removing polynomial trends: Suppose the data come from the model $Y_t = \mu_t + X_t$ where μ_t is a polynomial of order k and X_t is stationary, then differencing of order k: $\nabla^k Y_t = (I - B)^k Y_t$ results in stationary data to which an ARMA model can be fit.

2. Random walk models: Suppose that the data come from the random walk model: $Y_t = Y_{t-1} + X_t$ where X_t is stationary. Then clearly $\nabla Y_t = X_t$ is stationary and an ARMA model can be fit to this difference data.

Such models, which after appropriate differencing, reduce to ARMA models are called ARIMA models.

Definition 2.1 (ARIMA)**.** *A process Y_t is said to be ARIMA(p, d, q) with mean μ if $X_t = (I - B)^d Y_t$ is ARMA(p, q) with mean μ. In other words:*

$$\phi(B)(X_t - \mu) = \theta(B) Z_t,$$

where $\{Z_t\}$ is white noise.

3 Fitting ARIMA models

Just use the function *arima(dataset, order = c(p, d, q))*. I suggest you always use this function. If you know that you want to fit a pure AR model, you might consider the *ar()* function.

The arima function will give you the estimates of μ (under the name intercept), $\phi_1, \ldots, \phi_p$ and $\theta_1, \ldots, \theta_q$. It will also give you the estimated standard errors. An estimate of σ^2 is also provided.