Spring 2013 Statistics 153 (Time Series) : Lecture Eighteen

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1 The Sinusoid

Sinusoid: $R \cos(2\pi f t + \Phi)$. The following terminology is standard. R is called the *amplitude*, f is called the *frequency* and Φ is called the *phase*. The quantity $1/f$ is called the *period* and $2\pi f$ is termed the *angular frequency*. Note that three parameters R, f and Φ are involved in the definition of the sinusoid.

The function can also be written as $A \cos 2\pi f t + B \sin 2\pi f t$ where $A = R \cos \Phi$ and $B = R \sin \Phi$. This parametrization also has three parameters: f, A and B.

Yet another way of representing the sinusoid is to use complex exponentials:

$$\exp(2\pi i f t) = \cos(2\pi f t) + i \sin(2\pi f t).$$

Therefore

$$\cos(2\pi f t) = \frac{\exp(2\pi i f t) + \exp(-2\pi i f t)}{2} \text{ and } \sin(2\pi f t) = \frac{\exp(2\pi i f t) - \exp(-2\pi i f t)}{2i}.$$

Thus $A \cos 2\pi f t + B \sin 2\pi f t$ can also be written as a linear combination of $\exp(2\pi i f t)$ and $\exp(-2\pi i f t)$.

Sinusoids at certain special frequencies have nice orthogonality properties. Consider the vector:

$$u = (1, \exp(2\pi i/n), \exp(2\pi i 2/n), \dots, \exp(2\pi i(n-1)/n)).$$

This is the sinusoid $\exp(2\pi i f t)$ at frequency $f = 1/n$ evaluated at the time points $t = 0, 1, 2, \dots, (n-1)$.

Also define the vector corresponding to the sinusoid $\exp(2\pi i f t)$ at frequency $f = j/n$ evaluated at $t = 0, 1, \dots, (n-1)$ by

$$u^j = (1, \exp(2\pi i j/n), \exp(2\pi i 2j/n), \dots, \exp(2\pi i(n-1)j/n)).$$

These vectors $u^0, u^1, u^2, \dots, u^{n-1}$ are orthogonal. In other words, the dot product between u^k and u^l is zero if $k \neq l$. Therefore, every data vector $x := (x_1, \dots, x_n)$ can be written as a linear combination of $u^0, u^1, \dots, u^{n-1}$.

2 The Discrete Fourier Transform

Let $x_0, \dots, x_{n-1}$ denote data (real numbers) of length n.

The DFT of $\{x_t\}$ is given by $b_j, j = 0, 1, \dots, n-1$, where

$$b_j = \sum_{t=0}^{n-1} x_t \exp\left(-\frac{2\pi i j t}{n}\right) \qquad \text{for } j = 0, \dots, n-1.$$

Therefore, $b_0 = \sum x_t$. Also for $1 \le j \le n-1$,

$$b_{n-j} = \sum_t x_t \exp\left(-\frac{2\pi i(n-j)t}{n}\right) = \sum_t x_t \exp\left(\frac{2\pi ijt}{n}\right) \exp\left(-2\pi it\right) = \bar{b}_j.$$

Note that this relation only holds if $x_0, \ldots, x_{n-1}$ are real numbers.

Thus for $n = 11$, the DFT can be written as:

$$b_0, b_1, b_2, b_3, b_4, b_5, \bar{b}_5, \bar{b}_4, \bar{b}_3, \bar{b}_2, \bar{b}_1.$$

and for $n = 12$, it is

$$b_0, b_1, b_2, b_3, b_4, b_5, b_6 = \bar{b}_6, \bar{b}_5, \bar{b}_4, \bar{b}_3, \bar{b}_2, \bar{b}_1.$$

Note that b_6 is necessarily real because $b_6 = \bar{b}_6$.

The DFT is calculated by the R function `fft()`.

The original data $x_0, \ldots, x_{n-1}$ can be recovered from the DFT using:

$$x_t = \frac{1}{n} \sum_{j=0}^{n-1} b_j \exp\left(\frac{2\pi ijt}{n}\right) \qquad \text{for } t = 0, \ldots, n-1.$$

Thus for $n = 11$, one can think of the data as the 11 real numbers $x_0, x_1, \ldots, x_{10}$ or, equivalently, as one real number b_0 along with 5 complex numbers $b_1, \ldots, b_5$.

For $n = 12$, one can think of the data as the 12 real numbers $x_0, \ldots, x_{11}$ or, equivalenlty, as two real numbers, b_0 and b_6, along with the 5 complex numbers $b_1, \ldots, b_5$.

The following identity always holds:

$$n \sum_t x_t^2 = \sum_{j=0}^{n-1} |b_j|^2 .$$

Because $b_{n-j} = \bar{b}_j$, their absolute values are equal and we can write the above sum of squares identity in the following way. For $n = 11$,

$$n \sum_t x_t^2 = b_0^2 + 2|b_1|^2 + 2|b_2|^2 + 2|b_3|^2 + 2|b_4|^2 + 2|b_5|^2$$

and for $n = 12$,

$$n \sum_t x_t^2 = b_0^2 + 2|b_1|^2 + 2|b_2|^2 + 2|b_3|^2 + 2|b_4|^2 + 2|b_5|^2 + b_6^2.$$

Note that there is no need to put an absolute value on b_6 because it is real.

Because $b_0 = \sum_t x_t = n\bar{x}$, we have

$$n \sum_t x_t^2 - b_0^2 = n \sum_t x_t^2 - n^2 \bar{x}^2 = n \sum_t (x_t - \bar{x})^2.$$

Thus the sum of squares identity can be written for n odd, say $n = 11$, as

$$\sum_t (x_t - \bar{x})^2 = \frac{2}{n}|b_1|^2 + \frac{2}{n}|b_2|^2 + \frac{2}{n}|b_3|^2 + \frac{2}{n}|b_4|^2 + \frac{2}{n}|b_5|^2$$

and, for n even, say $n = 12$, as

$$\sum_t (x_t - \bar{x})^2 = \frac{2}{n}|b_1|^2 + \frac{2}{n}|b_2|^2 + \frac{2}{n}|b_3|^2 + \frac{2}{n}|b_4|^2 + \frac{2}{n}|b_5|^2 + \frac{1}{n}b_6^2.$$