Statistics 153 (Introduction to Time Series) Homework Four

Due on 24 April, 2013

14 April, 2013

1. **DFT and Convolution**: For two datasets $x_0, x_1, \ldots, x_{n-1}$ and $y_0, y_1, \ldots, y_{n-1}$, their *convolution* is the dataset $z_0, z_1, \ldots, z_{n-1}$ defined by

$$z_i = \sum_{j=0}^{n-1} x_{i-j} y_j \qquad \text{for } i = 0, \ldots, n-1$$

where $x_{-k} = x_{-k+n}$. Find the DFT of $z_0, \ldots, z_{n-1}$ in terms of the DFTs of $x_0, \ldots, x_{n-1}$ and $y_0, \ldots, y_{n-1}$.

2. **Chirps**:

   (a) Consider the data $x_t = \sin(\pi t^2/256)$ for $t = 0, 1, \ldots, 127$. Plot the data. Also plot the magnitude (absolute) of the DFT coefficients $b_1, \ldots, b_{64}$. Comment on the two plots.

   (b) Repeat the previous exercise for $y_t = \sin(\pi t^2/512)$ and $z_t = \sin(\pi t^2/1024)$. Comment on the differences between the plots.

3. **Leakage Reduction with a Different Window**: Consider the data, $x_t = 10\sin(2\pi f_0 t) + 3\cos(2\pi f_0 t)$ for $t = 0, \ldots, n-1$ with $n = 100$ and $f_0 = 0.062$ that we looked at in class. We have seen that the Leakage in the DFT is substantially reduced if we multiply the data (Hanning) by:

$$h_t = 1 - \cos(2\pi t/n) \qquad \text{for } t = 0, 1, \ldots, n-1.$$

As suggested in Bloomfield's book (Page 69), consider instead the following window function:

$$w_t = w_p\left(\frac{2t+1}{2n}\right),$$

where

$$\begin{aligned} w_p(x) &= 1 - \cos(2\pi x/p) \qquad \text{for } 0 \le x < p/2 \\ &= 1 \qquad \text{for } p/2 \le x < 1 - p/2 \\ &= 1 - \cos(2\pi(1-x)/p) \qquad \text{for } 1 - p/2 \le x \le 1. \end{aligned}$$

Multiply the data x_t by this window w_t for $p = 0.1$ and plot the magnitudes of the DFT terms $b_1, \ldots, b_{50}$ of the resulting data. How much has the leakage reduced compared to Hanning? How much has the leakage reduced compared to the DFT of the raw data? Repeat the exercise for $p = 0.2$ and $p = 0.5$. Comment on how the plot changes as p is increased.

4. Consider the sequence of random variables:

$$X_t = A\cos\left(\frac{9}{2}\pi t\right) + B\sin\left(\frac{9}{2}\pi t\right) + Z_t \qquad \text{for } t = \ldots, -2, -1, 0, 1, 2, \ldots$$

where A and B are uncorrelated random variables with mean 0 and variance σ^2 and $\{Z_t\}$ is white noise with mean 0 and variance σ_Z^2. Also assume that A and B are uncorrelated with the white noise $\{Z_t\}$.

(a) Show that $\{X_t\}$ is stationary and find its autocovariance function.

(b) Find the spectral distribution function of $\{X_t\}$.

(c) Does $\{X_t\}$ have a spectral density? Explain with reason.