1. **DFT and Convolution:** For two datasets $x_0, x_1, \ldots, x_{n-1}$ and $y_0, y_1, \ldots, y_{n-1}$, their convolution is the dataset $z_0, z_1, \ldots, z_{n-1}$ defined by

$$z_i = \sum_{j=0}^{n-1} x_{i-j} y_j \quad \text{for } i = 0, \ldots, n-1$$

where $x_{-k} = x_{k+n}$. Find the DFT of $z_0, \ldots, z_{n-1}$ in terms of the DFTs of $x_0, \ldots, x_{n-1}$ and $y_0, \ldots, y_{n-1}$.

2. **Chirps:**
   
   (a) Consider the data $x_t = \sin(\pi t^2/256)$ for $t = 0, 1, \ldots, 127$. Plot the data. Also plot the magnitude (absolute) of the DFT coefficients $b_1, \ldots, b_{64}$. Comment on the two plots.
   
   (b) Repeat the previous exercise for $y_t = \sin(\pi t^2/512)$ and $z_t = \sin(\pi t^2/1024)$. Comment on the differences between the plots.

3. **Leakage Reduction with a Different Window:** Consider the data, $x_t = 10 \sin(2\pi f_0 t) + 3 \cos(2\pi f_0 t)$ for $t = 0, \ldots, n-1$ with $n = 100$ and $f_0 = 0.062$ that we looked at in class. We have seen that the Leakage in the DFT is substantially reduced if we multiply the data (Hanning) by:

$$h_t = 1 - \cos(2\pi t/n) \quad \text{for } t = 0, 1, \ldots, n-1.$$

As suggested in Bloomfield’s book (Page 69), consider instead the following window function:

$$w_t = w_p \left( \frac{2t + 1}{2n} \right),$$

where

$$w_p(x) = \begin{cases} 
1 - \cos(2\pi x/p) & \text{for } 0 \leq x < p/2 \\
1 & \text{for } p/2 \leq x < 1 - p/2 \\
1 - \cos(2\pi (1 - x)/p) & \text{for } 1 - p/2 \leq x \leq 1.
\end{cases}$$

Multiply the data $x_t$ by this window $w_t$ for $p = 0.1$ and plot the magnitudes of the DFT terms $b_1, \ldots, b_{50}$ of the resulting data. How much has the leakage reduced compared to Hanning? How much has the leakage reduced compared to the DFT of the raw data? Repeat the exercise for $p = 0.2$ and $p = 0.5$. Comment on how the plot changes as $p$ is increased.
4. Consider the sequence of random variables:

\[ X_t = A \cos \left( \frac{9}{2} \pi t \right) + B \sin \left( \frac{9}{2} \pi t \right) + Z_t \quad \text{for } t = \ldots, -2, -1, 0, 1, 2, \ldots \]

where \( A \) and \( B \) are uncorrelated random variables with mean 0 and variance \( \sigma^2 \) and \( \{Z_t\} \) is white noise with mean 0 and variance \( \sigma^2_Z \). Also assume that \( A \) and \( B \) are uncorrelated with the white noise \( \{Z_t\} \).

(a) Show that \( \{X_t\} \) is stationary and find its autocovariance function.

(b) Find the spectral distribution function of \( \{X_t\} \).

(c) Does \( \{X_t\} \) have a spectral density? Explain with reason.