

Statistics 153 (Time Series) : Lecture Eight

Aditya Guntuboyina

09 February 2012

1 ARMA Crux

ARMA difference equation:

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q},$$

where $\{Z_t\}$ is white noise.

An equivalent representation of this difference equation is $\phi(B)X_t = \theta(B)Z_t$ with $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$.

Does there exist a stationary solution to $\phi(B)X_t = \theta(B)Z_t$? To answer this, find all roots of the polynomial $\phi(z)$ i.e., solve $\phi(z) = 0$. If **no root lies on the unit circle**: $\{z : |z| = 1\}$, then a stationary solution exists and is unique. Otherwise, no stationary solution exists.

Suppose there exists a stationary solution to $\phi(B)X_t = \theta(B)Z_t$. Is the solution causal? If **no root lies in the unit disk**: $\{z : |z| \leq 1\}$, then the stationary solution is causal. Otherwise, it is not causal. (Recall that X_t is said to be causal if it only depends on $Z_t, Z_{t-1}, Z_{t-2}, \dots$).

2 Calculation of Autocovariance of ARMA Processes

From now on, we shall consider only causal, stationary ARMA processes: $\phi(B)X_t = \theta(B)Z_t$.

2.1 First method

A causal, stationary ARMA process can be explicitly written as

$$X_t = \psi(B)Z_t = \psi_0 Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \dots \quad (1)$$

where $\psi(z) = \theta(z)/\phi(z)$. Note that ψ_0 will always equal one. This explicit representation can be used to calculate the acvf:

$$\gamma_X(h) = \text{cov}(X_t, X_{t+h}) = \sigma_Z^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \quad \text{for } h \geq 0, \quad (2)$$

where σ_Z^2 is the variance of the white noise process Z_t . The acf (autocorrelation function) can be calculated from this via $\rho_X(h) = \gamma_X(h)/\gamma_X(0)$.

This method requires the explicit evaluation of $\psi_0, \psi_1, \dots$.

As an example consider the first order moving average process: $X_t = Z_t + \theta Z_{t-1}$. This is of the form (1) with $\psi_0 = 1, \psi_1 = \theta$ and $\psi_j = 0$ for $j \geq 2$. Thus (2) with these ψ_j s gives

$$\gamma_X(0) = \sigma_Z^2(1 + \theta^2) \quad \gamma_X(1) = \sigma_Z^2\theta \quad \gamma_X(h) = 0 \quad \text{for } h \geq 2.$$

The corresponding autocorrelations are

$$\rho_X(1) = \frac{\theta}{1 + \theta^2} \quad \rho_X(h) = 0 \quad \text{for } h \geq 2.$$

Thus for MA(1), autocorrelations drop after the first lag. The autocorrelation at lag one has the same sign as θ and its maximum and minimum values are $1/2$ (for $\theta = 1$) and $-1/2$ (for $\theta = -1$) respectively.

For the second order MA process: $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$, we have $\psi_0 = 1, \psi_1 = \theta_1, \psi_2 = \theta_2$ and $\psi_j = 0$ for $j \geq 3$. The identity (2) with these ψ_j s gives

$$\gamma_X(0) = \sigma_Z^2(1 + \theta_1^2 + \theta_2^2) \quad \gamma_X(1) = \sigma_Z^2\theta_1(1 + \theta_2) \quad \gamma_X(2) = \sigma_Z^2\theta_2 \quad \gamma_X(h) = 0 \quad \text{for } h \geq 3.$$

The corresponding autocorrelations are given by

$$\rho_X(1) = \frac{\theta_1(1 + \theta_2)}{1 + \theta_1^2 + \theta_2^2} \quad \rho_X(2) = \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2} \quad \rho_X(h) = 0 \quad \text{for } h \geq 3.$$

Let us now look at AR(1) given by $X_t - \phi X_{t-1} = Z_t$. For this to be stationary and causal, we have seen that the condition $|\phi| < 1$ is necessary and sufficient. In this case, $X_t = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \dots$. This is same as (1) if we take $\psi_j = \phi^j$ for $j = 0, 1, 2, \dots$. Thus formula (2) for the autocovariance gives:

$$\gamma_X(h) = \sigma_Z^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+h} = \sigma_Z^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} = \sigma_Z^2 \frac{\phi^h}{1 - \phi^2} \quad \text{for } h > 0,$$

which results in the simple expression for the autocorrelation: $\rho_X(h) = \rho^h$ for $h > 0$.

For a general causal stationary ARMA process $\phi(B)X_t = \theta(B)Z_t$, one can write $\phi(z) = (1-a_1z)(1-a_2z)\dots(1-a_pz)$ where $1/a_1, \dots, 1/a_p$ are the (possibly complex) roots of the $a_1, \dots, a_p$ each satisfying $|a_i| < 1$ so that

$$\begin{aligned}\psi(z) &= \frac{\theta(z)}{\phi(z)} \\ &= \frac{\theta(z)}{(1-a_1z)\dots(1-a_pz)} \\ &= \theta(z)(1-a_1z)^{-1}\dots(1-a_pz)^{-1} \\ &= \theta(z)(1+a_1z+a_1^2z^2+\dots)(1+a_2z+a_2^2z^2+\dots)\dots(1+a_pz+a_p^2z^2+\dots).\end{aligned}$$

The product above can, in principle, be multiplied out to get the expression $\psi_0 + \psi_1z + \psi_2z^2 + \dots$. These ψ_j s can then be plugged into the formula (2) to obtain the autocovariance function of X_t . Of course, this might be very tedious in some cases.

2.2 Second Method

The second method works particularly well for AR processes (i.e., the moving average polynomial $\theta(z) = 1$) of order $p > 1$ where it might be hard to explicitly find the coefficients ψ_j in (2).

Consider a causal, stationary ARMA satisfying the difference equation:

$$X_t - \phi_1X_{t-1} - \dots - \phi_pX_{t-p} = Z_t + \theta_1Z_{t-1} + \dots + \theta_qZ_{t-q}.$$

Now fix $k \geq 0$ and take the covariance of both sides of the above equation with X_{t-k} . On the left hand side, we would get:

$$\text{cov}(\phi(B)X_t, X_{t-k}) = \gamma_X(k) - \phi_1\gamma_X(k-1) - \dots - \phi_p\gamma_X(k-p). \quad (3)$$

To evaluate the right hand side, we use the expression $X_t = \psi_0Z_t + \psi_1Z_{t-1} + \dots$ (valid only under causality) to get

$$\text{cov}(\theta(B)Z_t, X_{t-k}) = \text{cov}(Z_t + \theta_1Z_{t-1} + \dots + \theta_qZ_{t-q}, \psi_0Z_{t-k} + \psi_1Z_{t-k-1} + \psi_2Z_{t-k-2} + \dots)$$

If $k > q$, the above covariance is zero. And if $k \leq q$, then

$$\text{cov}(\theta(B)Z_t, X_{t-k}) = (\psi_0\theta_k + \psi_1\theta_{k+1} + \dots + \psi_{q-k}\theta_q)\sigma_Z^2. \quad (4)$$

Equating (3) and (4), we get

$$\gamma_X(k) - \phi_1\gamma_X(k-1) - \dots - \phi_p\gamma_X(k-p) = (\psi_0\theta_k + \psi_1\theta_{k+1} + \dots + \psi_{q-k}\theta_q)\sigma_Z^2, \quad (5)$$

for $0 \leq k \leq q$ and

$$\gamma_X(k) - \phi_1 \gamma_X(k-1) - \cdots - \phi_p \gamma_X(k-p) = 0 \quad \text{for } k > q. \quad (6)$$

The autocovariance function γ_X is obtained by solving these equations.

Example 2.1 (AR(1) Process). For AR(1), $p = 1$ so that left hand side in (5) and (6) becomes $\gamma_X(k) - \gamma_X(k-1)$. Also $q = 0$ and thus (5) which only holds for $k \leq q = 0$ gives only one equation (recall $\gamma_X(-h) = \gamma_X(h)$):

$$\gamma_X(0) - \phi \gamma_X(1) = \sigma_Z^2 \psi_0 \theta_0. \quad (7)$$

On the other hand, (6) holds for all $k > q = 0$ which implies

$$\gamma_X(k) - \phi \gamma_X(k-1) = 0 \quad \text{for all } k \geq 1. \quad (8)$$

This equation is known as the **Yule-Walker equation for order 1**.

Note that (8) allows us to deduce $\gamma_X(1) = \phi \gamma_X(0)$, $\gamma_X(2) = \phi \gamma_X(1) = \phi^2 \gamma_X(0)$ and, in general, $\gamma_X(k) = \phi^k \gamma_X(0)$ for $k \geq 1$. The equations (7) and (8) for $k = 1$ together imply $\gamma_X(k) = \sigma_Z^2 \psi_0 \theta_0 / (1 - \phi^2)$. Finally, observe that $\theta_0 = 1$ and $\psi_0 = 1$.

Example 2.2 (ARMA(1,1)). The difference equation is $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$. Just as in the AR(1) case, we need $|\phi| < 1$ to ensure stationarity and causality. We have $p = 1, \phi_0 = 1, \phi_1 = \phi$ and $q = 1, \theta_0 = 1, \theta_1 = \theta$.

The equations (5) and (6) give

$$\gamma_X(0) - \phi \gamma_X(-1) = (1 + \psi_1 \theta_1) \sigma_Z^2 \quad \text{for } k = 0,$$

$$\gamma_X(1) - \phi \gamma_X(0) = \theta \sigma_Z^2 \quad \text{for } k = 1,$$

and

$$\gamma_X(k) = \phi \gamma_X(k-1) \quad \text{for } k \geq 2.$$

The number ψ_1 is the coefficient of z in $(1 + \theta z)(1 + \phi z + \phi^2 z^2 + \dots)$ and equals $\theta + \phi$. Solving these equations, we get

$$\gamma_X(0) = \sigma_Z^2 \frac{1 + \theta^2 + 2\phi\theta}{1 - \phi^2} \quad \gamma_X(k) = \phi^{k-1} \sigma_Z^2 \frac{(\theta + \phi)(1 + \theta\phi)}{1 - \phi^2}.$$

This results in the autocorrelations:

$$\rho_X(k) = \frac{(\theta + \phi)(1 + \theta\phi)}{1 + \theta^2 + 2\phi\theta} \phi^{k-1} \quad \text{for } k \geq 1.$$

Example 2.3 (AR(2)). Consider the AR(2) model given by $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$. For this to be causal, stationary, we need the autoregressive

polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ to have both roots outside the unit disk. The roots are easily found and are given by:

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}$$

These roots are both outside the unit disk if and only if all three of the following conditions hold:

$$\phi_1 + \phi_2 < 1 \quad \phi_2 - \phi_1 < 1 \quad |\phi_2| < 1.$$