Recall that we are given \( v(S) \) for all sets \( S \subseteq [n] := \{1, \ldots, n\} \), with \( v(\emptyset) = 0 \), and \( v(S \cup T) \geq v(S) + v(T) \) if \( S, T \subseteq [n] \) are disjoint.

**Theorem 1** Shapley’s four axioms uniquely determine the functions \( \phi_i \).
Moreover, we have the random arrival formula:

\[
\phi_i(v) = \frac{1}{n!} \sum_{k=1}^{n} \sum_{\pi \in S_n, \pi(k)=i} \left( v(\pi(1), \ldots, \pi(k)) - v(\pi(1), \ldots, \pi(k-1)) \right)
\]

**Remark.** A probabilistic interpretation of this formula was presented at the end of the preceding lecture.

**Proof.** Recall the game for which \( w_S(T) = 1 \) if \( S \subseteq T \), and \( w_S(T) = 0 \) in the other case. The symmetry and dummy axioms give that \( \phi_i(w_S) = 1/|S| \) if \( i \in S \), with \( \phi_i(w_S) = 0 \) otherwise.

Our aim is, given \( v \), to find coefficients \( \{c_S\}_{S \subseteq [n], S \neq \emptyset} \) such that

\[
v = \sum_{\emptyset \neq S \subseteq [n]} c_S w_S.
\]  

Firstly, we will assume (1), and determine the values of \( \{c_S\} \). Applying (1) to the singleton \( \{i\} \):

\[
v(\{i\}) = \sum_{\emptyset \neq S \subseteq [n]} c_S w_S(\{i\}) = c_{\{i\}} w_i(i) = c_i,
\]  

where we may write \( c_i \) in place of \( c_{\{i\}} \). More generally, suppose that we have determined \( c_S \) for all \( S \) with \( |S| < l \). We want to determine \( c_{\tilde{S}} \) for some \( \tilde{S} \) with \( |\tilde{S}| = l \). We have that

\[
v(\tilde{S}) = \sum_{\emptyset \neq S \subseteq [n]} c_S w_S(\tilde{S}) = \sum_{S \subseteq \tilde{S}, |S| < l} c_S + c_{\tilde{S}}.
\]  

This determines \( c_{\tilde{S}} \). Now let us verify that (1) does indeed hold. Define the coefficients \( \{c_S\} \) via (2) and (3), inductively for sets \( \tilde{S} \) of size \( l > 1 \): that is,

\[
c_{\tilde{S}} = v(\tilde{S}) - \sum_{S \subseteq \tilde{S}, |S| < l} c_S.
\]
However, once (2) and (3) are satisfied, (1) also holds (something that should be checked by induction). We now find that

$$
\phi_i(v) = \phi_i\left( \sum_{\emptyset \neq S \subseteq [n]} c_{S w S} \right) = \sum_{\emptyset \neq S \subseteq [n]} \phi_i(c_{S w S}) = \sum_{S \subseteq [n], i \in S} \frac{c_S}{|S|}.
$$

This completes the proof of the first statement made in the theorem. As for the second: for each permutation $\pi$ with $\pi(k) = i$, we define

$$
\psi_i(v, \pi) = v(\pi(1), \ldots, \pi(k)) - v(\pi(1), \ldots, \pi(k - 1)),
$$

and

$$
\Psi_i(v) = \frac{1}{n!} \sum_\pi \psi_i(v, \pi).
$$

Our goal is to show that $\psi_i(v)$ is identically equal to $\phi_i(v)$. For a given $\pi$, note that $\psi_i(v, \pi)$ satisfies the dummy and efficiency axioms. It also satisfies the additivity axiom, but not the symmetry axiom.

We now show that averaging produces a new object that also satisfies the symmetry axiom - that is, that $\{\psi_i(v)\}$ satisfies this axiom. To this end, suppose that $i$ and $j$ are such that

$$
v(S \cup \{i\}) = v(S \cup \{j\})
$$

for all $S \subseteq [n]$ with $S \cap \{i, j\} = \emptyset$. For every permutation $\pi$, define $\pi^*$ that switches the locations of $i$ and $j$. That is, if $\pi(k) = i$ and $\pi(l) = j$, then $\pi^*(k) = j$ and $\pi^*(l) = i$, with $\pi^*(r) = \pi(r)$ with $r \neq k, l$. We claim that

$$
\psi_i(v, \pi) = \psi_j(v, \pi^*).
$$

Suppose that $\pi(k) = i$ and $\pi(l) = j$. Note that $\psi_i(v, \pi)$ contains the term

$$
v(\pi(1), \ldots, \pi(k)) - v(\pi(1), \ldots, \pi(k - 1)),
$$

whereas $\psi_i(v, \pi^*)$ contains the corresponding term

$$
v(\pi^*(1), \ldots, \pi^*(k)) - v(\pi^*(1), \ldots, \pi^*(k - 1)).
$$

We find that

$$
\psi_i(v) = \frac{1}{n!} \sum_{\pi \in S_n} \psi_i(v, \pi) = \frac{1}{n!} \sum_{\pi \in S_n} \psi_j(v, \pi^*)
$$

$$
= \frac{1}{n!} \sum_{\pi^* \in S_n} \psi_j(v, \pi^*) = \psi_j(v),
$$

where in the second equality, we used the fact that the map $\pi \rightarrow \pi^*$ is a one-to-one map from $S_n$ to itself, for which $\pi^{**} = \pi$. This completes the
proof of the second statement in the theorem. □

**A simple example.** A seller has a fish having no intrinsic value to him. A buyer values the fish at 10 dollars. We find the Shapley value: suppose that the buyer pays $x$ for the fish, with $0 < x \leq 10$. Writing $S$ and $B$ for the seller and buyer, we have that $v(S) = 0$, $v(B) = 0$, with $v(S, B) = (10 - x) + x$, so that $\phi_s(v) = \phi_B(v) = 5$.

A potential problem with using the Shapley value in this case is the possibility that the buyer under-reports his desire for the fish to the party that arbitrates the transaction.

**Example.** Find the Shapley values for the following variant of the glove game. There are $n = r + 2$ players. Players 1 and 2 have left gloves. The remaining players each have a right glove. Note that $V(S)$ is equal to the maximal number of proper and disjoint pairs of gloves. In other words, $V(S)$ is equal to the minimum of the number of left, and of right, gloves held by members of $S$. Note that $\phi_1(v) = \phi_2(v)$, and $\phi_r(v) = \phi_3(v)$, for each $r \geq 3$. Note also that

$$2\phi_1(v) + r\phi_3(v) = 2,$$

provided that $r \geq 2$. For which permutations does the third player add value to the coalition already formed? The answer is the following orders:

$$13, 23, \{1, 2\}3, \{1, 2, j\}3,$$

where $j$ is any value in $\{4, \ldots, n\}$, and where the curly brackets mean that each of the resulting orders is to be included. The number of permutations corresponding to these possibilities is: $r!$, $r!$, $2(r - 1)!$, and $6(r - 1) \cdot (r - 2)!$. This gives that

$$\phi_3(v) = \frac{2r! + 8(r - 1)!}{(r + 2)!}.$$

That is,

$$\phi_3(v) = \frac{2r + 8}{(r + 2)(r + 1)r}.$$