Obstacle Courses

1. Introduction.

Here are two situations involving chance:

(i) Someone rolls a die three times. (People usually roll dice in pairs, so “dice” is more common than “die”, the singular form.)

(ii) A box contains six tickets marked as shown:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

Someone takes out three tickets, one at a time, at random, from the box, leaving three tickets behind in the box.

In (i), it could happen the die comes up:

\[
\begin{array}{c}
\bullet, \text{ then } \star, \text{ then } \star \\
\end{array}
\]

And in (ii), it could happen the tickets come out:

\[
\begin{array}{c}
1, \text{ then } 2, \text{ then } 3 \\
\end{array}
\]

Neither possibility is very likely. But do they have the same chance? Before reading on, you might look up from the page and try to answer the question. No experience in calculating chances is required, only a little reasoning.
To get 1, 2, 3 from the box (in that order), the following must happen:

the 1 must be drawn from 1 2 3 4 5 6

and then

the 2 must be drawn from 2 3 4 5 6

and then

the 3 must be drawn from 3 4 5 6

The chances are not the same. The box is more likely than the die to lead to the 1, 2, 3 because on the second and third draws, the chance of getting the right number from the box is larger than the one in six chance of rolling the right number with the die.

Part of the reasoning involved three small mental steps, each consisting of pinning down exactly what is in the box before another ticket is drawn. This kind of reasoning comes up often in chance calculations, and it helps to have a diagram to guide you through the steps. The idea behind the diagram is as follows. Think of the box and tickets as hoping they are going to produce the 1, 2, 3:

We all want

1 then 2 then 3!

You know three things stand in their way. On the first draw, the 1 must come out of the box. If another number shows up, the box and tickets will have their hopes dashed. So that makes one obstacle:

1 first

After that, the box and tickets face another obstacle. The next ticket must be a 2:

1 first 2 next
And there is one obstacle left:

That does part of the diagram. To finish it, you have to write a chance under each obstacle. Take the obstacles one at a time. The first one is straightforward. You want to find the chance the box and tickets make it over the first obstacle. For that to happen, the 1 must be drawn first, and the chance of that is 1/6. Write that under the first obstacle:

The second obstacle will take more effort. First—and this is the key step in arriving at the correct chance—imagine that the box and tickets made it over the first obstacle. In this example, if the box and tickets got over the first obstacle, then the 1 is no longer in the box, leaving the other five tickets for the second draw:

Now you are ready to find the chance for the second obstacle. The box and tickets made it over the first obstacle, and now they want to get over the second one. And for that to happen, the 2 must come up:

The chance is 1/5. Put that fraction under the second obstacle:

For the third obstacle, follow the same reasoning. First—the key step
again—imagine that the box and tickets made it over the first two obstacles. Then the 1 and 2 are gone from the box, leaving the other four tickets for the third draw:

```
3 4 5 6
```

The box and tickets made it over the first two obstacles, and now they want to get over the third one. For that to happen, the 3 must come up on the third draw:

```
All we need is 3 and we'll be over the obstacles!
```

```
3 4 5 6
```

The chance is 1/4, and it goes under the last obstacle:

```
1 first 2 next 3 last
1/6 1/5 1/4
```

That completes the diagram for the possibility:

```
1 first, then 2, then 3
```

The diagram is an *obstacle course*. Obstacle courses are useful when a chance situation proceeds in stages, and you want to know the chance of something happening at the first stage and something else happening at the second stage and something else again at the third stage, and so on. In the example, there were three stages—the three draws from the box—and the chance being considered is that of getting 1 at the first stage and 2 at the second and 3 at the third.

With one exception, the examples below ask you to set up an obstacle course for a possibility. A good way to read an example is to read the question, but cover up the answer with a piece of paper. Then try to get the obstacle course on your own. If you can't get started, uncover just a couple of lines of the answer to get a hint. If you think you have got the right answer, again just look at the first couple of lines of the answer to see if you are headed in the right direction. And so on.
**Example 1.**

A box contains six tickets marked as shown:

```
[S O B E I T]
```

Someone takes out three tickets, one at a time, at random, from the box, leaving three tickets behind in the box.

It could happen that:
- the 1st ticket drawn has a letter from SO,
- the 2nd has a letter from BE,
- and the 3rd has a letter from IT.

Write down an obstacle course for this possibility.

**Answer.**

There are three draws where the right ticket must come up, so sketch three obstacles:

```
\[ \bigg/ \bigg/ \bigg/ \]
```

Next, write at the top of each obstacle what is required to get over it:

```
\text{letter from SO first} \quad \text{letter from BE next} \quad \text{letter from IT last}
```

The chance the first ticket has either S or O on it is 2/6, and that's the chance of getting over the first obstacle. So 2/6 belongs under the first obstacle:

```
\[ \bigg/ \bigg/ \bigg/ \quad 2/6 \quad \bigg/ \bigg/ \bigg/ \]
```

To get the chance for the second obstacle, pretend the box and tickets made it over the first obstacle. Then there are five tickets in the box, and two of them are marked with the letters (B or E) that will get the box and tickets over the second obstacle. The chance for the second obstacle is 2/5:

```
\[ \bigg/ \bigg/ \bigg/ \quad 2/6 \quad \bigg/ \bigg/ \bigg/ \quad 2/5 \quad \bigg/ \bigg/ \bigg/ \]
```

Finally, pretend the box and tickets made it over the first two ob-
stacles. Then there are four tickets in the box, and two of them are marked with the letters (I or T) that will get the box and tickets over the third obstacle. Put 2/4 beneath the third obstacle to finish the diagram:

\[
\begin{array}{ccc}
\text{letter from} & \text{letter from} & \text{letter from} \\
\text{SO first} & \text{BE next} & \text{IT last} \\
2/6 & 2/5 & 2/4 \\
\end{array}
\]

End of answer.

Example 2.
The setup is the same as in Example 1; the same box and again three draws:

\[
\begin{array}{cccc}
S & O & B & E & I & T \\
\end{array}
\]

It could happen that:
- one of the tickets (1st or 2nd or 3rd) has a letter from SO,
- another ticket has a letter from BE,
- and the remaining one has a letter from IT.

Write down an obstacle course for this possibility.

Answer.
There are three obstacles:

\[
\begin{array}{ccc}
\swarrow & \swarrow & \swarrow \\
\end{array}
\]

The first ticket could have any letter on it:

\[
\begin{array}{ccc}
\text{1st letter} & \swarrow & \swarrow & \swarrow \\
\text{any letter} & & & \\
\end{array}
\]

The second ticket must have a letter not from the word containing the first letter:

\[
\begin{array}{ccc}
\text{1st letter} & \swarrow & \swarrow & \swarrow \\
\text{any letter} & \swarrow & \swarrow & \swarrow \\
\text{2nd letter not in word containing the 1st letter} & & & \\
\end{array}
\]

The caption for the third obstacle is shown at the top of the next page.
1st letter  2nd letter not in word  3rd letter not in words containing
any letter  containing the 1st letter  either 1st letter or 2nd letter

For the first obstacle, the chance is 6/6. For the second, there are five
tickets left in the box, and one to avoid; the chance is 4/5. For the
third, there are four tickets left in the box, and two to avoid:

1st letter  2nd letter not in word  3rd letter not in words containing
any letter  containing the 1st letter  either 1st letter or 2nd letter

6/6  4/5  2/4

End of answer.

The next example just reviews some basic chances for a deck of cards.
If you haven’t played any card games for a while, you might want to
read the following before going on.

A deck of playing cards contains 52 cards: 13 spades, 13 hearts, 13 diamonds, 13
clubs. Each group of 13 cards consists of an ace, a king, a queen, a jack; and nine
cards numbered: 10, 9, 8, 7, 6, 5, 4, 3, 2. The spades and clubs are black; the
hearts and diamonds are red

Example 3.
Someone is going to shuffle a deck of cards and deal out the top card
on the deck. The card might be:
(a) the queen of hearts.
(b) an ace.

What is the chance of each possibility?

Answer.
(a) For the queen of hearts to be dealt, it is going to have to be at the
top of the deck. However it could be anywhere in the deck, from the
top position to the bottom one. There are 52 positions in all, and
because the cards are shuffled, it is equally likely to be in any of
them. So the chance the queen of hearts ends up at the top of the
deck is 1 in 52. The answer is 1/52.
(b) For an ace to be dealt, the top card on the deck must be an ace. Each ace has 1 chance in 52 of being the top card in the deck, and since there are 4 aces in the deck, that makes 4 chances in 52 there is an ace at the top of the deck. The answer is 4/52. (This could be simplified to 1/13.)

End of answer.

Example 4.
The dealer in a card game is going to shuffle a deck of cards and then deal out, one at a time, the top three cards on the deck. It might turn out that the first card is an ace, the second is a king, and the third is an ace. Write down an obstacle course for this possibility.

Answer.
There are three stages, so sketch three obstacles:

Next, write at the top of each obstacle what is required to get over it:

Now for the chances at the base of the obstacles. The first obstacle requires that the first card dealt be an ace. The chance is 4/52 (example 2), and that goes under the first obstacle:

For the second obstacle, start—as always—by pretending that the deck of cards made it over the first obstacle. Then an ace is gone from the deck. For the deck to get over the next obstacle, the second card dealt must be a king. So there must be a king at the top of the 51 cards remaining. The chance of that is 4/51, because the 4 kings are still there in the 51 cards, and the 51 cards were themselves shuffled when the dealer shuffled the full deck. Put 4/51 under the second obstacle:

For the third obstacle, imagine that the deck made it over the first
two obstacles. Then two cards are gone from the deck, one an ace, the other a king. That leaves 50 cards in the deck, with 3 aces and 3 kings in the 50 cards. For the deck to get over the third obstacle, another ace must be dealt. So there must be an ace at the top of the 50 cards remaining. The chance of that is 3/50 because there are 3 aces in the 50 cards. The chance for the third obstacle is 3/50:

\[
\begin{align*}
\text{ace} & \quad \text{king} & \quad \text{ace} \\
\frac{4}{52} & \quad \frac{4}{51} & \quad \frac{3}{50}
\end{align*}
\]

That completes the obstacle course for the possibility that the first card dealt is an ace, the second a king, and the third an ace.

**Example 5.**

Ann and Bob each draw a ticket at random from the box shown below—Ann first, then Bob.

\[
\begin{array}{cccc}
A & A & * & B & B \\
\end{array}
\]

Ann wins a dollar if the ticket she draws is marked with the letter A or an * (asterisk); Bob wins a dollar if the ticket he draws is a B or an *. If you can, write down an obstacle course for the possibility that both Ann and Bob win a dollar. If you cannot, try to say why not.

**Answer.**

Part of the obstacle course is straightforward:

\[
\begin{align*}
\text{Ann wins} & \quad \text{Bob wins} \quad 3/5
\end{align*}
\]

To find the missing chance, the first step is to imagine that the box and tickets made it over the first obstacle. Then Ann must have got an A or the *. With that ticket gone, the box is either:

\[
\begin{array}{cccc}
A & * & B & B \\
\end{array} \quad \text{or} \quad \begin{array}{cccc}
A & A & B & B \\
\end{array}
\]

Bob will draw his ticket from one of these two, and the chance he wins a dollar—gets over the second obstacle—will be either 3/4 or 2/4, depending on the box. So, after getting over the first obstacle, there seem to be two chances to write at the bottom of the second obstacle. And there is no way to pick one over the other, because you don’t know which ticket Ann got, the A or the *. The starting point of the reasoning—that the box and tickets made it over the first obstacle—
does not give enough information to get the chance for the second obstacle. The possibility that both Ann and Bob win cannot be represented as an obstacle course.

Exercises for section 1.

In the exercises below, various possibilities are given. For each one, write down the obstacle course or explain why it cannot be done. No calculations are required, only reasoning. Answers are given in section 4.

1. A die is rolled four times.
   (a) A six comes up on the first roll.
   (b) A six only comes up on the first roll.
   (c) The first number rolled does not come up again.

2. Three cards are dealt from a deck of cards.
   (a) All three cards are hearts.
   (b) None of the three cards are hearts.

3. A box contains eight tickets marked as shown:

   \[
   \begin{array}{ccccccc}
   E & L & E & V & A & T & O & R \\
   \end{array}
   \]

   Two tickets will be taken out, one at a time, at random, leaving six tickets in the box.
   (a) The letters on the two tickets are vowels.
   (b) The first is a vowel and the second is an E.

4. As in 3, except that three tickets are drawn from the box.
   (a) The first letter is A, the next is R, and the last is T.
   (b) The letters left in the box are: E L E V O

5. A coin is tossed five times.
   (a) Heads come up only on the last two tosses.
   (b) Heads come up twice somewhere in the five tosses.

2. Calculations.

Once an obstacle course has been set up, it is a matter of routine to find the chance of getting over all the obstacles. Just multiply together the chances written under the obstacles.

*Example 1.*

Two possibilities are described below. Without doing any arithmetic,
you can see that one is a little less likely than the other. Which one?

(i) Someone tosses a coin five times in a row. Heads could come up every time.

(ii) Someone deals out five cards from a deck of cards. The cards could all be red.

Find the chances in (i) and (ii).

*Answer.*

Five red cards is less likely than five heads. If the first card is red, the chance the next card is also red is 25 out of 51, and the chances keep going down—although not by much—as more red cards are dealt. For the coin, the chances stay at 50-50.

The obstacle courses for (i) and (ii) are:

(i)

<table>
<thead>
<tr>
<th></th>
<th>heads</th>
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<tbody>
<tr>
<td></td>
<td>1/2</td>
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<td>red</td>
<td>red</td>
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<td>red</td>
<td>red</td>
<td>red</td>
</tr>
</tbody>
</table>

(ii)

<table>
<thead>
<tr>
<th></th>
<th>red</th>
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<th>red</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>26/52</td>
<td>25/51</td>
<td>24/50</td>
<td>23/49</td>
<td>22/48</td>
</tr>
</tbody>
</table>

To get the chance of five heads, multiply the five 1/2s to get 1/32, which is about a 3% chance. To get the chance of five reds, multiply the five fractions at the base of the obstacles to get (after cancelling common factors) 253/9996, which is around 2.5%—a little smaller than 3%. Both these chances are about the same as rolling box-cars (two sixes) with a pair of dice: 1/36 or 2.8%.

**Example 2.**

A hypothetical lottery—TINY TEN—works as follows. You buy a ticket that shows the numbers from one to ten:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

You circle three numbers. Then the lottery organization draws three tickets, one at a time, at random, from the box shown below. No tickets are put back in the box, so no number can be drawn twice.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

The numbers on the three tickets are the winning numbers. If the
numbers you circled on the ticket match the winning numbers, you win the lottery. For example, if you circled 3, 5, 9, and the first ticket was 9, the second was 3 and the third was 5, you would win. Order does not matter. What is the chance you win the lottery?

Answer.
The first step is to try to write down an obstacle course. Imagine watching the tickets as they are drawn from the box. Right away, the number on the first ticket must be one of the three numbers you circled. If it isn’t, you might as well throw away your ticket, because you have just missed one of the winning numbers. (For example, if you circled 3, 5, 9, and the number on the first ticket was 8, you have missed 8). For the same reason, the number on the second ticket must be another one of your choices. And finally, the number on the third ticket must be the one circled number remaining. That makes three obstacles standing between you and winning:

\[
\begin{align*}
\text{one of the circled numbers} & \quad \text{another circled number} & \quad \text{the last circled number} \\
\text{3/10} & \quad \text{2/9} & \quad \text{1/8}
\end{align*}
\]

The chance of getting over the first obstacle is 3/10. Next suppose you made it over the first obstacle. Then you have one winning number and two to go. To get a second one, the number on the next ticket must match one of your remaining two circled numbers. So the chance of getting over the second obstacle is 2/9. The same reasoning leads to 1/8 for the last obstacle.

\[
\begin{align*}
\text{one of the circled numbers} & \quad \text{another circled number} & \quad \text{the last circled number} \\
\text{3/10} & \quad \text{2/9} & \quad \text{1/8}
\end{align*}
\]

The chance of winning TINY TEN is 6/720 or 1 in 120.
Exercises for section 2.

1. A roulette wheel at Nevada has 38 pockets, in each of which the ball is equally likely to fall. Out of the 38 pockets, 18 are red, 18 are black, and 2 are green.
   (a) A Nevada wheel is spun 2 times. Find the chance the ball falls in a red pocket on the first spin and a black pocket on the second spin. Or more briefly, find the chance of RB (red then black).
   (b) A Nevada wheel is spun 3 times. Find the chance the ball falls in a red pocket on the first spin, a black pocket on the second spin, and a red pocket on the third spin. Or more briefly, find the chance of RBR.
   (c) A Nevada wheel is spun 20 times. Find the chance the ball never falls in a green pocket.

2. Five cards (a poker hand) are dealt, one at a time, from an ordinary deck of cards.
   (a) Find the chance that only aces or kings appear.
   (b) Find the chance that neither aces nor kings appear.
   (You do not need to work out the arithmetic in (a) or (b).)

3. A gambler puts money on the number 33 in the game of Keno. This is a single number bet, and you can think of it this way. A box contains eighty tickets, numbered from 1 to 80. Twenty tickets are drawn at random from the box, leaving sixty tickets in the box. If 33 is one of the 20 numbers drawn, the gambler wins a prize. Otherwise, he loses. Find the chance the gambler loses.

4. A word is selected at random from the sentence: YOU BEAT ME TO THE PUNCH and then a letter is selected at random from that word. Which letter seems more likely to be chosen, “M” or “N”? Which letter seems more likely to be chosen, “B” or “Y”? (Try to answer without calculating.) Find the chances of choosing M, N, B, and Y.

5. Someone shuffles a deck of cards and proceeds to turn the cards over, one at a time, until the first time a diamond shows up. Then she will stop.
   (a) Find the chance she stops at the 4th card.
   (b) Find the chance she stops at the 4th card, and that card is the ace of diamonds.
   (You do not need to work out the arithmetic in (a) or (b).)
3. Comment.

This section is optional. It goes a little further into the ideas behind the method of obstacles.

Example 1.
A mini-deck of cards contains only ten cards: nine are red and one is black. The mini-deck is shuffled and the top three cards are dealt. Find the chance the three cards are red.

Answer.
There are three obstacles:

1st card red \[\frac{9}{10}\]  
2nd card red \[\frac{8}{9}\]  
3rd card red \[\frac{7}{8}\]

The chance that all three cards are red is \[\frac{7}{10}\].

The method of obstacles for finding a chance takes place in two steps: set up an obstacle course, and then multiply together the chances written under the obstacles. Multiplication is familiar, but setting up an obstacle course requires reasoning that is new to most people. Take the obstacle course in the example. To get the \[\frac{8}{9}\] under the second obstacle you have to imagine that the mini-deck got over the first obstacle. (If it had not, the first card would be black and the chance would be 1 that the second card is red.) But the mini-deck might not make it over the first obstacle. So the \[\frac{8}{9}\] used in the multiplication is based on something that did not have to happen. The same is true of the \[\frac{7}{8}\]. To get that, you have to imagine the deck made it over the first two obstacles. That is, the first two cards are red. That doesn’t have to happen either.

The method of obstacles involves a long chain of these “ifs”. If a deck of cards (say) made it over the first obstacle, the chance of making it over the second obstacle would be such and such, and if the deck made it over first two obstacles, the chance of making it over the third one would be—and so on. Then the chances arising from the “ifs” are multiplied together. Reasoning involving a lot of “ifs” is not natural for the human mind. As the “ifs” pile up, mental fatigue sets in, and we start to doubt the conclusion. But the method gets the chance right. The mini-deck, with only ten cards, is so simple that you can check the answer by other reasoning. One card is black, so the only
way the three cards dealt could be red is if the black card is somewhere in the bottom 7 cards of the deck. The chance of that is 7/10 because the black card, as a result of the shuffle, is equally likely to end up at any of the ten positions in the mini-deck.

4. Answers to exercises of section 1.

1. A die is rolled four times.
   (a) A six comes up on the first roll.
      
      \[
      \begin{array}{cccc}
      \text{1st is 6} & \text{2nd any} & \text{3rd any} & \text{4th any} \\
      \frac{1}{6} & \frac{6}{6} & \frac{6}{6} & \frac{6}{6}
      \end{array}
      \]

   (b) A six only comes up on the first roll.
      
      \[
      \begin{array}{cccc}
      \text{1st is 6} & \text{2nd not 6} & \text{3rd not 6} & \text{4th not 6} \\
      \frac{1}{6} & \frac{5}{6} & \frac{5}{6} & \frac{5}{6}
      \end{array}
      \]

   (c) The first number rolled does not come up again.
      
      \[
      \begin{array}{cccc}
      \text{1st is any} & \text{2nd not 1st} & \text{3rd not 1st} & \text{4th not 1st} \\
      \frac{6}{6} & \frac{5}{6} & \frac{5}{6} & \frac{5}{6}
      \end{array}
      \]

2. Three cards are dealt from a deck of cards.
   (a) All three cards are hearts.
      
      \[
      \begin{array}{ccc}
      \text{1st is a heart} & \text{2nd is a heart} & \text{3rd is a heart} \\
      \frac{13}{52} & \frac{12}{51} & \frac{11}{50}
      \end{array}
      \]

   (b) None of the three cards are hearts. (The “not H” is short for “not a heart”.)
      
      \[
      \begin{array}{ccc}
      \text{1st is not H} & \text{2nd is not H} & \text{3rd is not H} \\
      \frac{39}{52} & \frac{38}{51} & \frac{37}{50}
      \end{array}
      \]
3. (a) The letters on the two tickets are vowels.

\[
\begin{array}{ccc}
\text{1st letter is} & \text{2nd letter is} \\
\text{a vowel} & \text{a vowel} \\
\frac{4}{8} & \frac{3}{7}
\end{array}
\]

(b) The first is a vowel and the second is an E.

\[
\begin{array}{ccc}
\text{1st letter is} & \text{2nd letter is} \\
\text{a vowel} & \text{an E} \\
\frac{4}{8} & \frac{?}{?}
\end{array}
\]

The possibility cannot be represented as an obstacle course. Knowing that the box and tickets made it over the first obstacle doesn’t give you enough information to specify the chance of getting over the second obstacle. The chance might be 1/7 or 2/7, depending on the particular letter drawn first. If the first letter is “E”, the chance is 1/7; “A” or “O”, 2/7. But you only know the first letter is a vowel, not which letter it is.

4. (a) The first letter is A, the next is R, and the last is T.

\[
\begin{array}{ccc}
\text{1st letter} & \text{2nd letter} & \text{3rd letter} \\
is A & is R & is T \\
\frac{1}{8} & \frac{1}{7} & \frac{1}{6}
\end{array}
\]

(b) The letters left in the box are: E L E V O

\[
\begin{array}{ccc}
\text{1st letter is} & \text{2nd letter is one} & \text{3rd letter is the} \\
A, R, or T & \text{of the two letters} & \text{last letter out} \\
\text{still in the box} & \text{out of A, R, T} & \text{of A, R, T still} \\
\text{in the box} & \frac{3}{8} & \frac{2}{7} & \frac{1}{6}
\end{array}
\]

5. (a) Heads come up only on the last two tosses.

\[
\begin{array}{cccccc}
\text{1st is} & \text{2nd is} & \text{3rd is} & \text{4th is} & \text{5th is} \\
tails & tails & tails & heads & heads \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array}
\]
(b) Heads come up twice somewhere in the five tosses.
    This cannot be represented as an obstacle course because
    nothing is required to happen at any particular toss.

Comment: Some of your answers might not be the same as above. That doesn’t mean they are wrong. For example, in 1(a) you might have left out the last three obstacles. That answer would be correct. The answer with the four obstacles makes clear the difference between the possibility in (a) and the possibility in (b). In 1(c) you might have left out the first obstacle. That’s a good answer, too.