Homework 7

This problem is about generalized linear models involving the gamma distribution, which is useful for modelling non-negative random variables. There are several equivalent parametrizations of the gamma density. One of the common ones is

$$f(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}$$

Then

$$E(Y) = \frac{\alpha}{\lambda} \qquad Var(Y) = \frac{\alpha}{\lambda^2}$$

Sometimes it is more convenient to parametrize the distribution in terms of its mean $\mu = \alpha/\lambda$. Then

$$f(y) = \frac{(\alpha/\mu)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\alpha y/\mu}$$

Then $E(Y) = \mu$ and $Var(Y) = \mu^2/\alpha$. The ratio of the standard deviation to the mean is called the "coefficient of variation." In this case the coefficient of variation is $\alpha^{-1/2}$.

For the classical linear model the standard assumption is that the variance is constant and the mean changes as a function of covariates. The analogue for a generalized linear model based on a gamma is that the coefficient of variation is constant while the mean changes; that is, the standard deviation is proportional to the mean.

Suppose then that the mean, μ, is modelled as a function of $x\beta$, $g(\mu) = x\beta$ and it is assumed that α is constant.

1. Show that the gamma distribution belongs to the exponential family. Identify θ, $b(\theta)$, etc.

2. Consider the canonical link function, $g(\mu) = \mu^{-1}$. Suppose that the observations are (Y_i, x_i), $i = 1, \ldots, n$, and the Y_i are independent.

3. Give an expression for the log-likelihood.

4. Using (3) show that the maximizing β does not depend on α, and that the maximum likelihood estimate of β satisfies a system of equations of the form, $X^T(Y - \mu(\beta)) = 0$.

5. Derive the form of the update $h^{(k)} = \beta^{(k+1)} - \beta^{(k)}$.

6. Give the form of the IRLS algorithm and identify the adjusted dependent variable. Show that it is equivalent to (5).

7. What equation does the maximum likelihood estimate of α satisfy?

8. An un-natural aspect of the link function above is that $x\beta$ must be positive. For this and other reasons it may be desirable to consider a link function which automatically guarantees that $g(x\beta) > 0$. For the choice $g(\mu) = \exp(x\beta)$, explain how IRLS can be used to estimate β.