Homework 2

1. Suppose that in the standard linear model, the design matrix X contains a column of ones. Show that the sum of the residuals from the least squares fit equals 0.

   The sum of the residuals is the inner product of the residuals and the vector of 1's and these two vectors are orthogonal.

2. Suppose that the independent variables in a standard least squares model are replaced by rescaled variables $u_{ij} = k_j x_{ij}$ (for example, centimeters are converted to meters.) Show that $\hat{Y}$ does not change. Does $\hat{\beta}$ change? (Hint: express the new design matrix in terms of the old one.)

   Let $U = XK$ where $K = \text{diag}(k_1, \dots, k_p)$; $X = UK^{-1}$. Then

   $$\begin{aligned} \hat{Y} &= X(X^T X)^{-1} X^T Y \\ &= UK^{-1}(K^{-1}X^T X K^{-1})^{-1} K^{-1} U^T Y \\ &= U(U^T U)^{-1} U^T Y \end{aligned}$$

   The expression on the last line above is that for $\hat{Y}$ in the new model, so it doesn't change Similarly

   $$\begin{aligned} \hat{\beta}_{\text{new}} &= (U^T U)^{-1} U^T Y \\ &= K^{-1} \hat{\beta}_{\text{old}} \end{aligned}$$

   So $\hat{\beta}$ does change.

3. In order to estimate two parameters θ and ϕ a number of independent measurements are taken, each having errors with mean zero and variance σ^2:

   (a) n observations have mean θ.

   (b) m observations have mean $\theta - \phi$.

   (c) m observations have mean $\phi - \theta$.

   Set this up in the form of a standard linear model. What are the least squares estimates of θ and ϕ? Find $Var(\hat{\phi})$, $Cov(\hat{\phi}, \hat{\theta})$, and $Var(\hat{\phi} - \hat{\theta})$. What is the estimate of σ^2?

Let $\beta = (\theta \;\; \phi)^T$. The first n rows of X are $(1 \;\; 0)$; the next m rows are $(1 \;\; -1)$ and the last m are $(-1 \;\; 1)$. Then

$$(X^TX)^{-1} = \frac{1}{2mn}\begin{pmatrix} 2m & 2m \\ 2m & n+2m \end{pmatrix}$$

$$\begin{aligned} \hat{\theta} &= \frac{Y_{1\cdot}}{n} \\ \hat{\phi} &= \frac{Y_{1\cdot}}{n} - \frac{Y_{2\cdot}}{2m} + \frac{Y_{3\cdot}}{2m} \end{aligned}$$

where $Y_{k\cdot}$ is the sum of the Y's in the kth block of X.

The covariance matrix of $\hat{\theta}$ and $\hat{\phi}$ is $\sigma^2(X^TX)^{-1}$. Thus

$$\begin{aligned} Var(\hat{\theta}) &= \frac{\sigma^2}{n} \\ Cov(\hat{\theta}, \hat{\phi}) &= \frac{\sigma^2}{n} \\ Var(\hat{\phi} - \hat{\theta}) &= \sigma^2(-1 \;\; 1)(X^TX)^{-1}(-1 \;\; 1)^T \\ &= \frac{\sigma^2}{2m} \end{aligned}$$

$$s^2 = \frac{1}{n+2m-2}\sum(Y_i - \hat{Y}_i)^2$$

where $\hat{Y}_i$ is $\hat{\theta}$ for the first n, $\hat{\theta} - \hat{\phi}$ for the next m and $\hat{\phi} - \hat{\theta}$ for the last m observations.

4. For a linear model with $\Sigma_{ee} = \sigma^2 V$, where V is a known positive definite matrix, we have seen how to form the generalized least squares estimates. How could σ^2 be estimated?

   Suppose $\Sigma_{ee} = \sigma^2 V$, and $V = KK^T$. Then

$$\begin{aligned} Z &= K^{-1}Y \\ &= K^{-1}X\beta + K^{-1}e \\ &= R\beta + \delta \end{aligned}$$

Z now follows a standard linear model; $\hat{Z} = R(R^T R)^{-1} R^T Z$ and

$$\begin{aligned} \hat{Y} &= K\hat{Z} \\ &= X(X^T X)^{-1} X^T V^1 Y \end{aligned}$$

Thus

$$\begin{aligned} s^2 &= \frac{1}{n-p} \|Z - \hat{Z}\|^2 \\ &= \frac{1}{n-p} \|K^{-1}Y - K^{-1}\hat{Y}\|^2 \\ &= \frac{1}{n-p} (Y - \hat{Y})^T K^{-T} K^{-1} (Y - \hat{Y}) \\ &= \frac{1}{n-p} (Y - \hat{Y})^T V^{-1} (Y - \hat{Y}) \end{aligned}$$