Homework 1

1. Give two examples of a 3×3 projection matrices that project onto one and two dimensional subspaces (thus four examples in all). Identify the subspaces.

2. Let $\mathbf{X}$ be a random n-vector and let $\mathbf{Y}$ be a random vector with $Y_1 = X_1$, $Y_i = X_i - X_{i-1}$, $i = 1, 2, \ldots, n$. Use matrix methods to solve the following:

   (a) If the X_i are independent random variables, find the covariance matrix of $\mathbf{Y}$.

   $\Sigma_{XX} = \text{diag}(\sigma_1^2, \ldots, \sigma_n^2)$ and $Y = AX$ where $a_{ij} = 1$ if $i = j$, $a_{ij} = -1$ if $i = j - 1$ and 0 otherwise. (I also gave credit if it was assumed that $\Sigma_{XX} = \sigma^2 I$.) That is, A is a lower triangular matrix with the diagonal equal to 1 and the next lower diagonal equal to -1. $\Sigma_{YY} = \sigma^2 A \Sigma_{XX} A^T$ and

$$A\Sigma_{XX}A^T = \begin{pmatrix} \sigma_1^2 & -\sigma_1^2 & 0 & 0 & \ldots & 0 \\ -\sigma_1^2 & \sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & \ldots & 0 \\ 0 & -\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_2^2 & \ldots & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \ldots & \sigma_{n-1}^2 + \sigma_n^2 \end{pmatrix}$$

   (b) If the Y_i are independent random variables, find the covariance matrix of $\mathbf{X}$.

   Solve for X in terms of Y to find that $B = A^{-1}$ is a lower triangular matrix of 1's. Then $B\Sigma_{YY}B^T$ has ij element equal to $\sum_{k=1}^{\min(i,j)} \sigma_k^2$.

3. Let $X_1, X_2, \ldots, X_n$ be independent random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Let

$$Q = \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2$$

   Use matrix methods to find $E(Q)$ and show that $Q/2(n-1)$ is an unbiased estimate of σ^2.

   Express $Q = ||AX||^2 = X^T A^T AX = X^T BX$. A is an $(n-1) \times n$ matrix with $a_{ij} = -1$ if $i = j$, $a_{ij} = 1$ if $i = j - 1$ and 0 otherwise. B

is then a $n \times n$ matrix with $b_{11} = b_{nn} = 1$ and $b_{ii} = 2$ otherwise. Thus

$$\begin{aligned} E(Q(X)) &= \mu^2 \mathbf{1}^T B \mathbf{1} + trace(B\Sigma_{XX}) \\ &= \mu^2 ||A\mathbf{1}||^2 + \sigma^2 trace(B) \\ &= 0 + 2(n-1)\sigma^2 \end{aligned}$$