

Some Special Results of Measure Theory

By L. Le Cam¹

U.C. Berkeley

1. Introduction

The purpose of the present report is to record in writing some results of measure theory that are known to many people but do not seem to be in print, or at least seem to be difficult to find in the printed literature.

The first result, originally proved by a consortium including R.M. Dudley, J. Feldman, D. Fremlin, C.C. Moore and R. Solovay in 1970 says something like this: Let X be compact with a Radon measure μ . Let f be a map from X to a metric space Y such that for every open set $S \subset Y$ the inverse image, $f^{-1}(S)$ is Radon measurable. Then, if the cardinality of $f(X)$ is not outlandishly large, there is a subset $X_0 \subset X$ such that $\mu(X \setminus X_0) = 0$ and $f(X_0)$ is separable. Precise definition of what outlandishly large means will be given below.

The theorem may not appear very useful. However, after seeing it, one usually looks at empirical measures and processes in a different light. The theorem could be stated briefly as follows: A measurable image of a Radon measure in a complete metric space is Radon. Section 6 Theorem 7 gives an extension of the result where maps are replaced by Markov kernels. Section 8, Theorem 9, gives an extension to the case where the range space is paracompact instead of metric.

The second part of the paper is an elaboration on certain classes of measures that are limits in a suitable sense of “molecular” ones, that is measures carried by finite sets. It ties together several possible formulations of relations of measures and integrals of uniformly continuous functions. It also puts Prohorov’s theorem: relative compactness is equivalent to tightness, in

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a very different light. Typically, the conditions it gives for relative compactness are enormously weaker (in appearance at least) than tightness. It would be good to see whether they can be used instead of tightness to prove relative compactness.

Using some of the results of that second part, it is easy to see that several properties of measures on paracompact spaces can be checked on their continuous images in metric spaces. This allows us to replace the term “metric” by “paracompact” in the first part and gives many other results.

The results of the second and third part were established in 1968-69 by L. Le Cam. There were also related results of Berezanskii, [1968] of Granirer [1967] and several other authors. The results were further refined by Z. Frolik. In the intervening years they may have become obsolete. We have not checked. They are reproduced here because the proofs are reasonably simple and because they might prod other investigations. For other applications of similar results see the thesis of Errol Caby [1976].

2. Measurable images of Radon measures

Let X be a compact set. A positive Radon measure on X is a curious object, constructed from positive linear functionals on the space $C(X)$ of continuous functions on X . One considers a positive linear functional, denoted $\langle \mu, f \rangle = \int f d\mu$ for $f \in C(X)$. Then one extends the definition to lower semicontinuous functions by $\int f d\mu = \sup\{\int \gamma d\mu; \gamma \leq f, \gamma \in C(X)\}$ and finally to all functions f that for, each $\epsilon > 0$, can be squeezed between an upper semicontinuous g_ϵ and a lower semicontinuous h_ϵ so that $g \leq f \leq h$ and $|\int h_\epsilon d\mu - \int g_\epsilon d\mu| \leq \epsilon$, with $\int g_\epsilon = -\int(-g_\epsilon)$. This gives a functional defined on a large space of functions on X . The sets called “measurable” for μ are those whose indicator has an integral defined by that process.

Note that part of the definition of μ is its domain. One might be able to extend μ even further, but such extensions would not be called “Radon”. An illustration is given by a famous theorem of Kakutani and Oxtoby [1950]: Let λ be the (usual) Lebesgue measure on $[0, 1]$. It is a Radon measure. There is an extension ν of it that has the following properties: a) It is invariant by all one to one pointwise transformations of $[0, 1]$ that left λ invariant and b) its domain is such that the Hilbert space of ν -square integrable functions it generates has a basis of cardinality $2^{\mathfrak{C}}$.

Here \mathfrak{c} is the cardinality of the continuum. The Lebesgue measure itself give a Hilbert space with a countable basis. The measure ν extends λ to a much larger class of sets, though not to all subsets of $[0, 1]$. However it is not a Radon measure.

A characteristic of positive Radon measures is that the measure of a set is the supremum of the measures of the compact sets contained in it. That is, if μ is Radon and A is μ -measurable then for each $\epsilon > 0$ there is a compact $K_\epsilon \subset A$ with $\mu(A \setminus K_\epsilon) < \epsilon$. Besides Radon measures, we shall need some information on “measurable cardinals”. A measurable cardinal is a set S that admits a nontrivial probability measure μ defined for all subsets of S and taking only two values, so that $\mu(A) = [\mu(A)]^2$. By “nontrivial” is meant that each point has measure zero.

To see how large a measurable cardinal must be, let us first look at the first infinite cardinal \aleph_0 , cardinal of an infinite countable set. It has two remarkable properties as follows:

- A) If $n < \aleph_0$ then $2^n < \aleph_0$
- B) A set S of cardinality \aleph_0 cannot be written as a union $\bigcup_{j \in J} A_j$ where $\text{card } A_j < \aleph_0$ and $\text{card } J < \aleph_0$.

A cardinal \aleph is said to be (strongly) inaccessible if it is uncountable and has property (A). It is weakly inaccessible if it is uncountable and has property (B).

Now property (A) is very strong. A (strongly) inaccessible cardinal $\aleph > \aleph_0$ must also be larger than $\mathfrak{c} = 2^{\aleph_0}$ and $2^{\mathfrak{c}}$ and so forth for any infinite cardinal $n < \aleph$. That means that if one restricts oneself to accessible cardinals one can carry out all the usual set operations such as unions, products, powers (because $2 \leq m \leq n$ implies $m^n = 2^n$ for infinite n). One will never get any inaccessible cardinal by such operations. In other words, one can ignore inaccessible cardinals for all usual and statistical purposes.

As to measurability, it has been proved by Ulam and Tarski that a measurable cardinal must be inaccessible. Thus if one lives in the land or universe of accessible cardinals one can say that measurable cardinals are “outlandishly large”, giving a precise meaning to the sentence used in the Introduction.

Property (B) is not as strong as (A). Its implications may be weaker, depending on what axioms one adds to the usual set theory ones. Here the

usual axioms will mean the axioms of Zermelo-Fraenkel with the axiom of choice, referred to as ZFC.

In a system such as ZFC, or the Bernays-Gödel system, one can list the infinite cardinals in increasing order using the ordinals as indices for the list. It would look like $\aleph_0 < \aleph_1 < \dots < \aleph_w < \aleph_{w+1} < \dots$.

In such a list it is always true that $\text{card } \alpha \leq \aleph_\alpha$. However equality is not ruled out. One can readily obtain ordinals such that $\text{card } \alpha = \aleph_\alpha$. For instance define a sequence of pairs $(w(n), Z_n)$ as follows. Let $Z_1 = \aleph_0$ and let $w(1)$ be the first ordinal that has cardinality \aleph_0 . If $(w(n), Z_n)$ has been constructed, let $w(n+1)$ be the first ordinal that has cardinality Z_n and let $Z_{n+1} = \aleph_{w(n+1)}$. The limit $\alpha = \sup_n w(n)$ is such that $\aleph_\alpha = \sup_n Z_n$ is $\text{card } \alpha$. However that \aleph_α is clearly the sum of a countable set of cardinals $Z_n < \aleph_\alpha$. (One could continue further, say up to the first uncountable ordinal Ω if each time one gets a limit such as α and \aleph_α with $\text{card } \alpha = \aleph_\alpha$ one replaces (α, \aleph_α) by $(\alpha, \aleph_{\alpha+1})$ in the list). Now if α is a limit ordinal and if \aleph_α is weakly inaccessible then \aleph_α must equal the cofinality of α . The cofinality of α is the smallest cardinality of a set of ordinals $w_\xi < \alpha$ such that $\sup_\xi w_\xi = \alpha$.

Such a set cannot be obtained by the recursive process described above. It is a fact of life that the existence of such alephs cannot be proved in ZFC.

Now what has that to do with measurability? Define a weakly measurable cardinal as one of a set S that admits a probability measure μ defined on all its subsets giving mass zero to all points. This is the same as before except that $\mu(A)$, $A \subset S$ is allowed to be any value in $[0, 1]$ instead of being just zero or one as in our previous definition. In a world where there are no strongly inaccessible cardinals a weakly measurable cardinal would be one that admits an atomless probability measure defined on all its subsets.

Now Ulam (1930) has proved that if \aleph_α is not weakly measurable $\aleph_{\alpha+1}$ is not weakly measurable either. So $\aleph_1, \aleph_2, \dots, \aleph_w, \dots, \aleph_\Omega \dots$ are not weakly measurable. In fact Ulam's result imply that either a weakly measurable set is already (two-valued) measurable, or there is some weakly measurable cardinal $n \leq \mathfrak{c} = 2^{\aleph_0}$. That second option is ruled out by the ordinary continuum hypothesis $\mathfrak{c} = \aleph_1$. Thus the first option is consistent with ZFC. However one could also assume that on the contrary \mathfrak{c} admits an atomless probability measure defined for all its subsets. This will be responsible for some of the precautions in the statements given below

Now here is the promised theorem.

Theorem 1 (Consortium) *Let X be a compact space. Let μ be a positive Radon measure on X . Let f be a map from X to a metric space Y . Assume that for every open set $G \subset Y$ the inverse image $f^{-1}(G)$ is μ -measurable (that is, in the domain of the Radon measure μ).*

Assume also that the cardinal of $f(X)$ does not admit any nontrivial two valued probability measure defined for all subsets. Then there is a subset X_0 of X such that $\mu(X \setminus X_0) = 0$ and such that $f(X_0)$ is separable.

Remark. As we shall see in the proof, the result becomes much easier if one assumes that the cardinal of $f(X)$ is not even weakly measurable. Then the Radon structure of μ is not essential. We shall elaborate later on what this means for possible definitions of “distributions” for such items as ordinary empirical cumulatives. Because of this the preceding Theorem 1, although peculiar, is not entirely uninformative.

Proof. One can without loss of generality assume $f(X) = Y$. Let $\{G_j; j \in J\}$ be a covering of Y by open sets G_j . Assume that the index set J is well ordered. For any $j \in J$ let $U_j = \bigcup_i \{G_i; i < j\}$ and $A_j = G_j \setminus U_j$. This is an intersection of an open with a closed set. Deleting the empty A_j , one can assume that each A_j is nonempty. Let $B_j; j \in J$ be other subsets of Y . Form the set $\bigcup_j (A_j \cap B_j)$. This is called the result of operation (\mathcal{M}) on the B_j . It is a theorem of D. Montgomery (see for instance Kuratowski-Topologie, vol I pge 267) that if the B_j are Borel subsets of Y of one of the transfinite classes called F_α or G_α so is $\bigcup_j (A_j \cap B_j)$. In particular any union $\bigcup (A_j; j \in S)$ where S is an arbitrary subset of J is a Borel set. Now the measure μ has an image $f(\mu)$ on Y , but by writing $\nu(S) = \mu\{f^{-1}[\bigcup (A_j; j \in S)]\}$ one obtains a measure defined on all the subsets of the index set J . This is the image of μ by the map $g \circ f$ where g maps Y into J by $g(y) = j$ if $y \in A_j$.

Let us look at the measure ν on J . It may have atoms. That is there may be sets $S \subset J$ such that $\nu(S) > 0$ but such that for any subset S' of S one has either $\nu(S') = 0$ or $\nu(S') = \nu(S)$. Take such an atom S_0 and delete the rest of J . This gives a measure ν_0 carried by S_0 taking only two values and defined for all subsets of S_0 . Since we have assumed that the cardinality of $f(X) = Y$ is not measurable, the same applies to J (since we have deleted the empty A_j 's), hence also to S_0 . Thus the measure ν_0 carried by S_0 must be carried by a particular point $s_0 \in S_0$. Doing this for each atom of ν obtains a countable set $C \subset J$ that carries the atomic part of ν . Map back C into

X taking $(g \circ f)^{-1}(C) = D$, say. This is a μ -measurable set.

Now replace μ by μ_1 defined by $\mu_1(V) = \mu(V \cap D^c)$. This is still a Radon measure on X and its image by $g \circ f$ has no atoms. Let us first deal with this particular μ_1 . If it is not zero one can assume for simplicity that it is a probability measure so that its image ν_1 by $g \circ f$ is also a probability measure. Since ν_1 is atomless, one can divide J into two sets, say $D_{1,0}$ and $D_{1,i}$ so that $\nu_1(D_{1,i}) = \frac{1}{2}$. Dividing each set in equal parts one gets sets $D_{2,i}$; $i = 0, 1, 2, 3$ such that $\nu_1(D_{2,i}) = \frac{1}{2^2}$. Proceed on dividing each set into equal parts each time. It is clear that this gives a map, say φ , of J into the interval $[0, 1]$. For the Borel subsets of $[0, 1]$ the image λ_1 of ν_1 coincides with the Lebesgue measure λ . However λ_1 is defined on all the subsets of $[0, 1]$.

Now consider the composition of map $w = \varphi \circ g \circ f$. It is a map from X to $[0, 1]$. By construction it is “measurable” at least in the sense that the inverse image of any Borel subset of $[0, 1]$ is a μ_1 -measurable set in X .

This implies in particular that w is also “measurable” in the sense of Bourbaki’s definition of that term. Specifically, for every $\epsilon > 0$ there exists a compact subset K_ϵ of X such that $\mu_1(X \setminus K_\epsilon) < \epsilon$ and such that when restricted to K_ϵ the map w is *continuous*. Now consider an arbitrary subset $T \subset [0, 1]$. Its inverse image $w^{-1}(T)$ is μ_1 -measurable. Thus, as above, for every $\epsilon > 0$ there is a compact $K_\epsilon \subset w^{-1}(T)$, such that $\mu_1[w^{-1}(T) \setminus K_\epsilon] < \epsilon$ and such that when restricted to K_ϵ the map w is continuous. If so the image $w(K_\epsilon)$ is also a compact. It is such that $w(K_\epsilon) \subset T$ and that $\lambda_1[T \setminus w(K_\epsilon)] < \epsilon$. This is true for every $\epsilon > 0$. Thus T must be Lebesgue measurable. Since there do exist non Lebesgue measurable sets, we have reached a contradiction.

This leads to the conclusion that the measure μ_1 must in fact be equal to zero. Equivalently one can say that for every arbitrary covering $\{G_j; j \in J\}$ and partition $A_j = G_j \setminus [\cup(G_i; i < j)]$ the image of μ by f is carried by a countable subfamily of the A_j , or equivalently again, by a countable subfamily of the G_j . In particular one can take a cover by open balls $B_{j,m}$ of center $y_{j,m}$ and radius $\frac{1}{m}$. Here j is in a certain set J_m which may be highly infinite. However the set $J_{m,1}$ of these j ’s such that $\mu[f^{-1}(B_{j,m})] > 0$ is countable. Consider the subset of Y formed by the centers $y_{j,m}$ for which $j \in J_{m,1}$. Taking all these sets for all m one obtains a countable subset of Y . Let W be the closure of that set. It has the property that for each integer m the union W_m of the open balls of radius $1/m$ centered at elements of W contains all the measure for the image $f(\mu)$. Thus W , intersection of the W_m must also have full measure for $f(\mu)$. The inverse image $X_0 = f^{-1}(W)$

satisfies the conditions of the theorem. Hence the result. \square

Remark 1. If instead of assuming that the cardinal of $f(X)$ is non measurable (for two-valued measures) one had instead assumed that it cannot carry arbitrary atomless probability measures, the proof could have been simplified. One could remove the part of it that disposed of the purely non atomic μ_1 and avoid the reference to Lebesgue measure and μ -measurability in the Bourbaki sense.

Remark 2. One could also state the conclusion of the theorem by saying that if Y is complete the image $f\mu$ extends to a Radon measure.

Remark 3. The theorem could have been stated in a different form: If f satisfies the conditions of Theorem 1, it is already μ -measurable in the sense of Bourbaki. That is, for each $\epsilon > 0$ there is a compact K_ϵ with $\mu(X \setminus K_\epsilon) < \epsilon$ and with f continuous when restricted to K_ϵ .

Remark 4. Theorem 1 says something applicable to empirical measures or cumulatives. To see this, let λ be the Lebesgue measure on $X = [0, 1]$. Define a map f from $[0, 1]$ to the space \mathcal{B} of bounded functions on $[0, 1]$ as follows. If $x \in [0, 1]$ then $f(x)$ is the function $t \rightsquigarrow F_x(t)$ such that $F_x(t) = 0$ for $t < x$ and $F_x(t) = 1$ for $t \geq x$. Metrize the space \mathcal{B} by its sup norm

$$\|u\| = \sup_t |u(t)|.$$

In this case the image $f(X)$ is a discrete subset of \mathcal{B} . If $x_1 \neq x_2$ then $\|f(x_1) - f(x_2)\| = 1$. Any Lebesgue set of positive measure must have the cardinality \mathbf{c} of the continuum. Since $\mathbf{c}, 2^{\mathbf{c}}, \dots$ are all non measurable, Theorem 1 implies that there must be closed subsets S of $f(X)$ such that $f^{-1}(S)$ is not λ -measurable. This is not too surprising. Every subset of $f(X)$ is closed. The map f from X to $f(X) = Y \subset \mathcal{B}$ is one to one and the sets that map back to Lebesgue measurable sets are the images of these Lebesgue sets. If instead of taking just one observation on $[0, 1]$ for λ one would take many, say n , from some joint distribution μ on $[0, 1]^n$ and form the corresponding cumulative distribution in \mathcal{B} the conclusion would be the same. Assuming that μ is not purely atomic, there will be closed subsets of \mathcal{B} whose inverse image is not μ -measurable. This statement assumes that μ on $[0, 1]^n$ is a Radon measure so that its measurable sets differ from Borel sets by negligible sets.

Now take $[0, 1]^n$ for our set X and take for $f(x)$ the corresponding cumulative as element of the space \mathcal{B} of bounded functions on $[0, 1]^n$. In this way the “distribution” of the cumulative may be thought of as the image $f(\mu)$ of μ on \mathcal{B} . It is defined on some σ -field \mathcal{F} of subsets of \mathcal{B} (or of $Y = f(X)$). By Theorem 1 this σ -field cannot contain the Borel field of $f(X)$. However Theorem 1 does not say anything about the possibility of *extending* the definition of $f(\mu)$ to the Borel field. This is a different question. Here the image $Y = f(X)$ is again a discrete space with the cardinality \mathfrak{c} of the continuum. all its subsets are closed.

The question of possible extension of $f(\mu)$ then boils down to the following: Does the continuum \mathfrak{c} admit an atomless probability measure defined for all subsets.

If there was such a measure say P it would not be hard to match it with $f(\mu)$ where this is defined. This is easier to see if $n = 1$ and μ is Lebesgue on $[0, 1]$. Using the partitioning by half already used in the proof of Theorem 1 to construct our map φ , one can readily match the σ -field \mathcal{F} on which $f(\mu)$ is defined with a corresponding σ -field of subsets of \mathfrak{c} , matching at the same time the respective measures. This is clear for Lebesgue on $[0, 1]$ and works in the same manner for any non atomic μ on $[0, 1]^n$ and any non atomic P on \mathfrak{c} .

Any extension of the non atomic $f(\mu)$ to all subsets of $Y = f(X)$ would necessarily be non atomic. Indeed \mathfrak{c} is nonmeasurable. Thus atoms would be points and these are already in the domain of $f(\mu)$.

Thus the question becomes: Does there exist a purely non atomic probability measure P defined on all the subsets of \mathfrak{c} . In other words is \mathfrak{c} weakly measurable?

We have already pointed out that this depends on where \mathfrak{c} is located in the string of alephs. The answer is negative if \mathfrak{c} is strictly inferior to the first aleph, \aleph_{α_0} , such that α_0 has for cofinality \aleph_{α_0} itself. In such a case “weak measurability” and “measurability” would be equivalent (Ulam 1930).

Thus, for instance, if $\mathfrak{c} \leq \aleph_{\Omega}$ where Ω is the first uncountable ordinal, then our $f(\mu)$ on $f(X)$ does not admit extensions to all the Borel subsets of $f(X)$.

However, the location of \mathfrak{c} in the alephs depends on what axioms of set theory one is willing to assume, while Theorem 1 does not depend on such assumptions. (It depends, however on the existence of sets that are not Lebesgue measurable. These sets do not exist in some systems where the

axiom of choice is suitably restricted). We presume that most statisticians would cheerfully accept the hypothesis that \mathbf{c} is not weakly measurable, or even the hypothesis $\mathbf{c} < \aleph_{\alpha_0}$, even though this puts a crimp in what one can do with the distribution of empirical cumulatives.

Another implication of Theorem 1 can be described as follows. Take a stochastic process $Z : t \rightsquigarrow Z(t)$ defined on some non degenerate interval T of the line. Assume that the trajectories of the process are bounded, by, say, 0 and 1. The “distribution” of the process Z is then some measure m on some σ -field of subsets of $[0, 1]^T$. According to Kakutani [1943] there exist a Radon measure μ on the compact set $[0, 1]^T$ (for the topology of pointwise convergence) such that μ and m have the same values on sets defined by finitely or countably many Borel restrictions on the coordinates. Here the cardinality of $[0, 1]^T$ is clearly not a measurable one.

Thus, according to Theorem 1, if we map $X = [0, 1]^T$ into a metric space Y by a function f there will necessarily exist closed subsets of Y whose inverse images are not μ -measurable unless f has almost separable range. (This means that there is a subset X_0 of X with $\mu(X \setminus X_0) = 0$ and $f(X_0)$ separable, as in Theorem 1). Thus, for instance, take for Y the set $[0, 1]^T$ itself but with the supremum norm. Take for f the identity map. These will be closed subsets of Y that are not μ -measurable unless the measure is concentrated on a separable subset of Y . Thus, although the Kakutani extension is much richer than the usual product σ -field of Kolmogorov, it leaves out many sets. For examples of interesting sets that are not in the domain of the Kakutani extension see R.M. Dudley [1972] and [1989].

3. Molecular measures and their limits

Often probability spaces come with more structure than the standard triplet (Ω, \mathcal{F}, P) . For instance they may be metric or topological. It is then pleasant if the properties of the measure and the metric or topology are somewhat related. Here we study a class of measures that are nicely related to the uniform structure of the space on which they live. We have taken uniform spaces instead of topological spaces because they afford more flexibility. A uniform space is given by a set X and a filter \mathcal{V} of “vicinities” of the diagonal Δ of $X \times X$. This filter is assumed to be such that the diagonal Δ is included in every $V \in \mathcal{V}$. Also, if a $V \in \mathcal{V}$, its symmetric

$\{(x, y) : (y, x) \in V\}$ is also in \mathcal{V} . Finally for each $V \in \mathcal{V}$ there is a $W \in \mathcal{V}$ such that $W^2 \subset V$. Here W^2 is defined as the set of pairs (x, z) for which there is a y such that $(x, y) \in W$ and $(y, z) \in W$.

A pseudo metric ρ on X defines a uniform structure, by taking for basis of the vicinities the sets of the form $\{(x, y); \rho(x, y) < \epsilon\}$ for $\epsilon > 0$. Conversely it can be shown that every uniform structure can be generated in this manner by some family $\{\rho_\alpha; \alpha \in A\}$ of pseudo metrics.

A function f from a uniform space (X_1, \mathcal{V}_1) to another (X_2, \mathcal{V}_2) is called uniformly continuous if for every $V_2 \in \mathcal{V}_2$ there is a $V_1 \in \mathcal{V}_1$ such that $(f(x), f(y)) \in V_2$ for all $(x, y) \in V_1$.

A family $\{f_\alpha; \alpha \in A\}$ of such functions is called uniformly equicontinuous if for $V_2 \in \mathcal{V}_2$ there is a $V_1 \in \mathcal{V}_1$ such that $(f_\alpha(x), f_\alpha(y)) \in V_2$ for all pairs $(x, y) \in V_1$ and all $\alpha \in A$. The real line \mathbb{R} is a uniform space if one takes for base of the vicinities the sets $\{(x, y) : |x - y| < \epsilon\}$ for $\epsilon > 0$.

A separated uniform space is one in which if $x \neq y$ then there is a $V \in \mathcal{V}$ such that $(x, y) \notin V$. Separated uniform spaces admit uniformly continuous real valued functions in large quantities. Let us denote by D the space of all bounded real valued uniformly continuous functions on (X, \mathcal{V}) . It is a Banach space with dual D^* if one gives it the sup norm $\|f\| = \sup_x |f(x)|$. We shall be interested in various subspaces of D^* that are naturally linked to the uniform structure of (X, \mathcal{V}) . To describe them, let us introduce some particular subsets of D . A set $B \subset D$ will be called a UEB set if it is bounded and uniformly equicontinuous. "Bounded" means that there is some number $\alpha \in (0, \infty)$ such that $|f(x)| \leq \alpha$ for all $x \in X$ and all $f \in B$.

A set $S \subset X$ is called *precompact* if for every $V \in \mathcal{V}$ the set S can be covered by a finite family of sets $A_j, j = 1, \dots, n$ that are small of order V , that is such that $(x, y) \in V$ for every pair (x, y) of elements of A_j .

The set D can be given the uniform structure \mathcal{U} of uniform convergence on the precompact subsets of X . This is usually weaker than the structure generated by the sup norm.

The subspaces of D^* that will be of interest below are as follows

- 1) The space \mathcal{M}_s of linear functionals with finite support on X . Specifically a linear functional φ on D belongs to \mathcal{M}_s if there is a finite set $\{x_j; j = 1, \dots, n\}$ and coefficients $c_j \in \mathbb{R}$ such that $\langle \varphi, \gamma \rangle = \sum_j c_j \gamma(x_j)$ for all $\gamma \in D$. Because a mass c_j at x_j looks like an atom and because

φ is a finite sum of such atoms, the elements of \mathcal{M}_s have also been called *molecular measures*. See Berezanskii (1968).

- 2) The space \mathcal{M}_p of linear functionals whose restrictions to the unit ball $B_1 = \{\gamma : \gamma \in D, \|\gamma\| \leq 1\}$ are continuous for the precompact convergence \mathcal{U} .
- 3) The space \mathcal{M}_u closure of \mathcal{M}_s in D^* for the structure of uniform convergence on the UEB subsets of D .

This uniform structure will be called $[\mathcal{V}]$ to recall that it came from \mathcal{V} .

- 4) The space $\bar{\mathcal{M}}_u$ of elements of D^* whose restrictions to UEB subsets of D are continuous for U .

Note that we stay in D^* . Thus each μ in any of these spaces has a finite norm $\|\mu\| = \sup\{|\langle \mu, \gamma \rangle|; \gamma \in D, |\gamma| \leq 1\}$.

The spaces defined above clearly satisfy the relations

$$\mathcal{M}_s \subset \mathcal{M}_p \subset \mathcal{M}_u \subset \bar{\mathcal{M}}_u \subset D^*.$$

Another important property is given by the following result

Proposition 1 *Each one of the spaces listed above is a band in D^* . The spaces \mathcal{M}_u and $\bar{\mathcal{M}}_u$ always coincide. Balls of the type $\{\mu : \|\mu - \varphi\| \leq a\}$ and the positive cone \mathcal{M}_u^+ of \mathcal{M}_u are complete for the structure $[\mathcal{V}]$. If \mathcal{V} is metrizable, then \mathcal{M}_p is also equal to \mathcal{M}_u .*

This will be proved below in several steps.

One of the reasons for the importance of Proposition 1 is the completeness statement. It will allow us to use the compactness criteria of Grothendieck [1952]. See Section 9. The identity $\mathcal{M}_p = \mathcal{M}_u$ when (X, \mathcal{V}) is metrizable also allows another characterization of \mathcal{M}_u : Let (Y, \mathcal{V}_1) be another uniform space whose uniform structure is metrizable. Let f be a uniformly continuous map from (X, \mathcal{V}) to (Y, \mathcal{V}_1) . If $\mu \in \mathcal{M}_u$ for (X, \mathcal{V}) then its image $f(\mu)$ by f is in \mathcal{M}_p for (Y, \mathcal{V}_1) . It turns out that if $\mu \in D^*$ maps this way into \mathcal{M}_p for every metrizable (Y, \mathcal{V}_1) and every uniformly continuous map f then $\mu \in \mathcal{M}_u$. Now an element of \mathcal{M}_p on (Y, \mathcal{V}_1) is just something that extends to a Radon measure on the completion of (Y, \mathcal{V}_1) . This will also allow us to extend the result of Theorem 1 Section 2 to images of Radon measures into uniform

spaces: Suppose that K is compact and that μ is a Radon measure on K . Let f be a map from K to a uniform space (X, \mathcal{V}) . Assume that for every $\gamma \in D$ the function $\gamma \circ f$ is μ -integrable (in the Radon sense). Then, in a world that does not possess any measurable cardinals, the image of μ is in \mathcal{M}_u .

Note that we have not mentioned countable additivity of the measures. A simple example will show why. Let X_1 be the interval $[0, 1]$ of the line with its standard uniformity. Let $X \subset X_1$ be the set of rational numbers, with its standard uniformity \mathcal{V} inherited from X_1 . The set D of uniformly continuous functions on (X, \mathcal{V}) is the set of restrictions to X of continuous functions on X_1 . Every positive linear functional on the set $C(X_1)$ of continuous functions on X_1 can also be identified as a positive linear functional on D . Since in this example X is precompact, our space \mathcal{M}_u^+ coincides with the set of linear functionals that can be obtained that way. However if our arbitrary uniform space (X, \mathcal{V}) is *complete* the elements of \mathcal{M}_u are σ -smooth on D in the sense that if $\mu \in \mathcal{M}_u$ and if $u_n \in D$ decreases pointwise to zero on X , then $\langle \mu, u_n \rangle \rightarrow 0$. This will be a consequence of the results proved below.

Note however that \mathcal{M}_u does depend on the uniform structure \mathcal{V} on X , not only of the topology it generates. There are usually many uniform structures yielding the same topology. Two of them are particularly interesting. Assuming (X, \mathcal{V}) separated, for simplicity, one can define on X the smallest structure $\underline{\mathcal{V}}$ that make the elements of D uniformly continuous. Then the completion \bar{X} of X for $\underline{\mathcal{V}}$ is a compact set. The set D becomes the set of restrictions to X of the continuous functions $C(\bar{X})$ on \bar{X} . The corresponding $\mathcal{M}_u(X, \underline{\mathcal{V}})$ can be canonically identified to the Radon measures on \bar{X} .

Another interesting structure is $\bar{\mathcal{V}}$, the “universal structure” attached to (X, \mathcal{V}) . It is the one defined by all the pseudo-metrics ρ defined on X and such that $\rho(x, y)$ is (jointly) continuous. The corresponding set $D(X, \bar{\mathcal{V}})$ is the set $C^b(X)$ of bounded continuous functions on (X, \mathcal{V}) . The corresponding $\mathcal{M}_u(X, \bar{\mathcal{V}})$ is typically much smaller than $\mathcal{M}_u(X, \mathcal{V})$ and therefore smaller than $\mathcal{M}_u(X, \underline{\mathcal{V}})$.

If X is separated and complete for \mathcal{V} , its \mathcal{V} precompact sets have compact closure. Hence the structure \mathcal{U} becomes the structure of uniform convergence on compacts. Since compactness is a topological property a set that is compact for \mathcal{V} is also compact for $\bar{\mathcal{V}}$. By the Stone Weierstrass theorem $D(X, \mathcal{V})$ is dense for \mathcal{U} in the set $C^b(X)$ of bounded continuous functions on X . The elements of $\mathcal{M}_p(X, \mathcal{V})$ are also in $\mathcal{M}_p(X, \bar{\mathcal{V}})$. They extend to Radon measures

on X . However the conclusion does not extend to $\mathcal{M}_u(X, \mathcal{V})$.

Before leaving these generalities and studying more closely the structure of \mathcal{M}_u let us note the following. Start with an (X, \mathcal{V}) and define the corresponding \mathcal{M}_u with its uniformly $[\mathcal{V}]$ of uniform convergence on the UEB sets of D . The set X itself can be identified to a subset X' of \mathcal{M}_u by associating to each x the Dirac measure δ_x , probability measure concentrated at x . The structure induced by $[\mathcal{V}]$ on X' is the initial structure \mathcal{V} since \mathcal{V} can be defined by the family of pseudo-metric $\rho_{\mathcal{B}}(x, y) = \sup_{\gamma} \{|\gamma(x) - \gamma(y)| : \gamma \in \mathcal{B}\}$ as \mathcal{B} ranges through UEB sets. Indeed if $V \in \mathcal{V}$ is such that $\gamma \in \mathcal{B}$ implies $|\gamma(x) - \gamma(y)| \leq \epsilon$ for $(x, y) \in V$ then $\rho_{\mathcal{B}}(x, y) \leq \epsilon$ on V . Conversely if ρ is one of the pseudo-metrics used to define \mathcal{V} and $\rho \leq 1$, take the set \mathcal{B} of functions $\gamma \in D$ such that $|\gamma(x) - \gamma(y)| \leq \rho(x, y)$ and $|\gamma| \leq 1$. Then \mathcal{B} is UEB and $\rho(x, y) = \sup\{|\gamma(x) - \gamma(y)|; \gamma \in \mathcal{B}\}$. (Take $\gamma_z(x) = \rho(x, z); z \in X$).

This shows that the notation \mathcal{V} and $[\mathcal{V}]$ will not lead to confusion.

4. The structure of $\bar{\mathcal{M}}_u$.

In this section we consider a fixed separated uniform space (X, \mathcal{V}) with its space D of bounded uniformly continuous functions and the attached spaces of linear functionals $\mathcal{M}_s \subset \mathcal{M}_p \subset \mathcal{M}_u \subset \bar{\mathcal{M}}_u$.

Consider first the UEB subsets of D . If B_1 and B_2 are two such sets, so is their union B_1 and B_2 . The convex hull of a UEB set B is also UEB, so is the convex symmetrized hull

$$\{\gamma : \gamma = \sum c_j \gamma_j, \quad \sum |c_j| \leq 1, \quad \gamma_i \in B\}$$

In addition the pointwise closure of a UEB set is UEB. A pointwise closed UEB is compact for pointwise convergence and on it the topology of pointwise convergence coincides with the topology induced by the structure \mathcal{U} of uniform convergence on precompact sets. We shall also need the following simple observation

Lemma 1 *Let S be a UEB set that is compact for pointwise convergence on X . Then S is also compact for the weak topology $W(D, \bar{\mathcal{M}}_u)$.*

Proof. Take a compact (Hausdorff) space \mathcal{B} and let $C(\mathcal{B})$ be the space of continuous real functions on \mathcal{B} . Then \mathcal{B} is also compact for the weakest

topology that makes all $\gamma \in C(\mathcal{B})$ continuous because that topology is Hausdorff and certainly weaker than the initial one on \mathcal{B} and therefore identical to it. On our compact UEB set the elements of $\bar{\mathcal{M}}_u$ are continuous for the pointwise topology since it coincides with to \mathcal{U} -topology. \square

A consequence of this state of affairs is as follows.

Corollary 1 *The dual of $\bar{\mathcal{M}}_u$ for $[\mathcal{V}]$ is D . That is every linear functional defined on $\bar{\mathcal{M}}_u$ and $[\mathcal{V}]$ continuous is given by the evaluations $\langle \varphi, \gamma \rangle$ for $\varphi \in \bar{\mathcal{M}}_u$ and some $\gamma \in D$.*

Proof. This is well-known fact in the duality theory for locally convex spaces. Briefly if \mathcal{B} is a convex symmetric compact UEB then its second polar \mathcal{B}^{00} in the dual of $(\bar{\mathcal{M}}_u, [\mathcal{V}])$ is the closure of \mathcal{B} . However \mathcal{B} being compact is already closed.

It is clear from the definition that both \mathcal{M}_u and $\bar{\mathcal{M}}_u$ are closed subsets of D^* for the structure \mathcal{V} . Thus they are also closed for the stronger topology defined by the norm of D^* . The space \mathcal{M}_p is also closed for the norm topology. Note also the following property.

Lemma 2 *Let \mathcal{M} be any one of the spaces $\mathcal{M}_s, \mathcal{M}_p, \mathcal{M}_u$ or $\bar{\mathcal{M}}_u$. If μ is an element of \mathcal{M} then its positive part μ^+ also belongs to \mathcal{M} .*

Proof. One can define μ^+ by the relation $\langle \mu^+, \gamma \rangle = \sup_u \{ \langle \mu, u \rangle; 0 \leq u \leq \gamma, u \in D \}$ for any $\gamma \in D$ that is in the positive cone D^+ of D . Let $v \in D$ be such that $0 \leq v \leq 1$ and such that $\langle \mu^+, 1 \rangle \leq \langle \mu, v \rangle + \epsilon$. Now $\langle \mu, v \rangle + \epsilon$ may be written $\langle \mu^+, v \rangle - \langle \mu^-, v \rangle + \epsilon$, giving $\langle \mu^+, 1 - v \rangle + \langle \mu^-, v \rangle \leq \epsilon$. For any element γ of D one can write

$$\begin{aligned} \langle \mu^+, \gamma \rangle &= \langle \mu^+, v\gamma \rangle - \langle \mu^-, v\gamma \rangle + \langle \mu^-, v\gamma \rangle + \langle \mu^+, (1 - v)\gamma \rangle \\ &= \langle \mu, v\gamma \rangle + \langle \mu^-, v\gamma \rangle + \langle \mu^+, (1 - v)\gamma \rangle \\ &\leq \langle \mu, v\gamma \rangle + \epsilon \|\gamma\|. \end{aligned}$$

Define a new functional $v \times \mu$ by $\langle v \times \mu, \gamma \rangle = \langle \mu, v\gamma \rangle$. The foregoing inequality says that $\|v \times \mu - \mu^+\| \leq 2\epsilon$. Now if \mathcal{B} is an UEB set, so is the set $\{v\gamma; \gamma \in \mathcal{B}\}$. Thus $v \times \mu$ belongs to \mathcal{M}_u (resp $\bar{\mathcal{M}}_u, \mathcal{M}_p$) whenever μ does. It follows that μ^+ , limit of the $v \times \mu$ for the norm topology, is also in the same space. The case of \mathcal{M}_s is clear, hence the result. \square

Lemma 3 *The spaces \mathcal{M}_p , \mathcal{M}_u and $\bar{\mathcal{M}}_u$ are bands in D^* .*

Proof. Since all these spaces are closed in D^* for its norm topology and since Lemma 2 applies to them, it is sufficient to show that if $0 \leq \nu \leq \mu \in \mathcal{M}_p$ then $\nu \in \mathcal{M}_p$ and similarly for \mathcal{M}_u and $\bar{\mathcal{M}}_u$.

In all cases, if $0 \leq \nu \leq \mu$ and if $\epsilon > 0$, there is some $\gamma \in D$ such that $\|\nu - \gamma \times \mu\| < \epsilon$. This can be shown directly, see F. Riesz (1940) and Bochner and Phillips (1941) or L. Dubins (1969). Another procedure would be to complete X for the smallest uniform structure $\underline{\mathcal{V}}$ that makes the elements of D uniformly continuous. On the completion ν and μ become Radon measures such that $\nu \leq \mu$. Therefore there is a measurable f such that $0 \leq f \leq 1$ and $\nu = f \times \mu$. Then, for every $\epsilon > 0$ there is a $\gamma \in D$ such that its extension to the completion satisfies $\int |\gamma - f| d\mu < \epsilon$.

Thus since ν can be approximated as closely as one wishes in the norm by $\gamma \times \mu$ one concludes as in Lemma 2 that if $\mu \in \bar{\mathcal{M}}_u$ and $0 \leq \nu \leq \mu$ then $\nu \in \bar{\mathcal{M}}_u$. Similarly for \mathcal{M}_p or \mathcal{M}_u . Hence the result. \square

Theorem 2 *The positive cone $\bar{\mathcal{M}}_u^+$ of $\bar{\mathcal{M}}_u$ is the $[\mathcal{V}]$ closure in D^* of the positive cone \mathcal{M}_s^+ of \mathcal{M}_s . The space \mathcal{M}_u and $\bar{\mathcal{M}}_u$ are the same. The balls $\{\mu; \|\mu\| \leq b\}$ of \mathcal{M}_u are complete for $[\mathcal{V}]$ and so is \mathcal{M}_u^+ .*

Proof. Let $C = \mathcal{M}_s^+$ be the positive cone of \mathcal{M}_s and let \bar{C} be its closure for $[\mathcal{V}]$ in $\bar{\mathcal{M}}_u$. Let μ be a positive element of $\bar{\mathcal{M}}_u^+$. Suppose that $\mu \notin \bar{C}$. Then there is a $[\mathcal{V}]$ continuous linear functional γ and numbers $a < a + \epsilon$ such that

$$\langle \mu, \gamma \rangle = a < a + \epsilon \leq \langle \varphi, \gamma \rangle$$

for all $\varphi \in C$.

According to the corollary of Lemma 1, this linear functional γ is in fact an element of D . The inequality $a + \epsilon \leq \langle \varphi, \gamma \rangle$ for all $\varphi \in C$ implies that $\gamma \geq 0$, fairly obviously. Thus we have $a + \epsilon \leq \inf_{\varphi \in C} \{\langle \varphi, \gamma \rangle\} = 0$. This implies $\langle \mu, \gamma \rangle = a < 0$. This contradicts the positivity of μ and γ .

Thus $\mathcal{M}_u \supset \bar{\mathcal{M}}_u^+$ hence also $\bar{\mathcal{M}}_u = \bar{\mathcal{M}}_u^+ - \mathcal{M}_u$. Since $\mathcal{M}_u \subset \bar{\mathcal{M}}_u$, it follows that $\mathcal{M}_u = \bar{\mathcal{M}}_u$ as claimed.

For the completeness statement let S be the ball $S = \{\mu : \|\mu\| \leq b, \mu \in \mathcal{M}_u\}$. If $\mu \in S$ then μ^+ and μ^- are also in S . By the first part of the argument μ^+ is a $[\mathcal{V}]$ limit of elements $\varphi \in \mathcal{M}_s^+$ such that $\|\varphi\| \leq \mu^+$. A similar

statement applies to μ^- . Hence S is the $[\mathcal{V}]$ closure of the ball $\{\varphi : \|\varphi\| \leq b; \varphi \in \mathcal{M}_s\}$ in D^* . Now the corresponding ball of D^* is $w(D^*, D)$ compact, hence $w(D^*, D)$ complete, hence also $[\mathcal{V}]$ complete. Thus S being closed in that ball is also $[\mathcal{V}]$ complete. The argument for the completeness of \mathcal{M}_u^+ is the same. \square

Actually one can prove a better completeness result.

Theorem 3 *The space $\bar{\mathcal{M}}_u$ is complete for the structure of uniform convergence on the UEB subsets of D .*

Proof. Note that we have worked from the start with subsets of D^* , the space of bounded linear functionals on D . However according to an observation of E. Caby [1973] a linear functional whose restriction to UEB's is continuous for precompact convergence (or uniform convergence) is already bounded. Indeed any sequence $\{\gamma_n\}$, $\gamma_n \in D$ such that $\|\gamma_n\| \rightarrow 0$ is UEB.

Thus the set $\bar{\mathcal{M}}_u$ is exactly the space of (arbitrary) linear functionals whose restriction to UEB sets are continuous for \mathcal{U} . Such a space is complete for $[\mathcal{V}]$ according to a theorem of Grothendieck. (See Bourbaki, *Espaces vectoriels topologiques* Chp IV, section 3, exercise #3) \square

Note. The completeness statement in Theorem 3 may seem inconsequential. Nevertheless it will allow us to use the compactness criteria given by Grothendieck (1952). See Section 9.

Another property of \mathcal{M}_u can be stated as follows.

Proposition 2 *Let μ be a positive element of D^* . The condition $\mu \in \mathcal{M}_u$ is equivalent to the statement that every filter on $(D^*)^+$ that converges for $w(D^*, D)$ already converges for $[\mathcal{V}]$. On \mathcal{M}_u^+ the weak topology $w(\mathcal{M}_u, D)$ and the topology induced by $[\mathcal{V}]$ are the same.*

Proof. Let $\mu \in D^*$ be positive. It is the $w(D^*, D)$ limit of some filter \mathcal{F} on \mathcal{M}_s^+ . If \mathcal{F} converges also for $[\mathcal{V}]$ then $\mu \in \mathcal{M}_u$, by definition of \mathcal{M}_u . To prove the converse let S be a UEB subset of D . Let $\rho(x, y) = \sup\{|\gamma(x) - \gamma(y)| : \gamma \in S\}$ and let S_m be the set of functions γ such that $|\gamma| \leq m$ and $|\gamma(x) - \gamma(y)| \leq m\rho(x, y)$. Note that $S \subset S_1$ and that each S_m is also UEB.

Let μ be an element of \mathcal{M}_u^+ such that $\|\mu\| = 1$. Then for a given $\epsilon > 0$ and a given integer m there is some $\nu \in \mathcal{M}_s^+$ such that $\|\nu\| = 1$ and such that $|\langle \mu, \gamma \rangle - \langle \nu, \gamma \rangle| < \frac{\epsilon}{8}$ for every $\gamma \in S_m$. By definition of \mathcal{M}_s , this ν has

a finite support, say F . Let G be the set $G = \{x : \rho(x, F) > \frac{1}{m}\}$. Define a function f by $f(x) = \rho(x, G) [\rho(x, F) + \rho(x, G)]^{-1}$. This f is such that $0 \leq f \leq 1$. It is unity on F and zero on G . Also $\rho(x, F) + \rho(x, G) \geq \frac{1}{m}$ and a simple computation shows that $f \in S_m$. Since by construction $f(x) = 1$ for $x \in F$ one has $\langle \nu, f \rangle = 1$, hence $\langle \mu, f \rangle \geq 1 - \epsilon/8$.

Now take a filter \mathcal{F} on $[D^*]^+$ and assume that \mathcal{F} converges to μ for $w(D^*, D)$. There is then a set $A \in \mathcal{F}$ such that $|\langle \varphi, f \rangle - \langle \mu, f \rangle| < \epsilon/8$ for all $\varphi \in A$. Thus $\langle \varphi, f \rangle \geq 1 - \epsilon/4$ for all $\varphi \in A$.

Returning to the finite set F and the set S_1 one can find a finite family $\{\gamma_j : j = 1, \dots, n\}$ of functions $\gamma_j \in S_1$ such that $\inf_j \sup_x \{|\gamma(x) - \gamma_j(x)| : x \in F\} < \epsilon/4$ for every $\gamma \in S_1$. Now by definition of S_1 and ρ , the inequality $\sup_x \{|\gamma(x) - \gamma_j(x)| : x \in F\} < \epsilon/4$ implies $\sup_x \{|\gamma(x) - \gamma_j(x)| : x \in G^c\} < \epsilon/4 + \frac{2}{m}$. From this it follows that for each $\gamma \in S_1$ one has $\inf_j \|f\gamma - f\gamma_j\| \leq \frac{\epsilon}{4} + \frac{2}{m}$. By assumption $\lim_{\mathcal{F}} \langle \varphi, f\gamma_j \rangle = \langle \mu, f\gamma_j \rangle$ for each j . This implies

$$\limsup_{\mathcal{F}} \sup_{\gamma \in S_1} |\langle \varphi, f\gamma \rangle - \langle \mu, f\gamma \rangle| \leq \frac{\epsilon}{4} + \frac{2}{m}$$

and finally

$$\limsup_{\mathcal{F}} \sup_{\gamma \in S_1} |\langle \varphi, \gamma \rangle - \langle \mu, \gamma \rangle| \leq \epsilon + \frac{2}{m}.$$

Since ϵ and m are arbitrary this implies

$$\lim_{\mathcal{F}} \sup_{\gamma \in S_1} |\langle \varphi, \gamma \rangle - \langle \mu, \gamma \rangle| = 0,$$

hence the result. \square

The construction of a pseudo metric ρ from a UEB set and of functions such as f above will occur again in Section 5 below.

5. Functionals defined on a sublattice.

The main result of the present section is Theorem 1 stated below after a few preparatory lemmas. What it says in effect is that, if the structure \mathcal{V} of the set (X, \mathcal{V}) is metrizable, a bounded linear functional continuous for the structure \mathcal{U} of precompact convergence on UEB sets of D is already \mathcal{U} -continuous on the ball of D .

To prove it we start by examining properties of certain lattices of numerical functions on X . Here the algebraic and lattice operations refer to operations carried out *pointwise* on the functions. The basic object will be a set S of numerical functions defined on a set X and subject to the following assumptions

- (A1) The constant functions $\gamma \equiv a$ belong to S for every $a \in [0, 1]$ and $\gamma \in S$ implies $0 \leq \gamma \leq 1$.
- (A2) The set S is a lattice for the operations $\gamma_1 \wedge \gamma_2$ and $\gamma_1 \vee \gamma_2$ carried out pointwise.
- (A3) The set S is convex.
- (A4) If a is a number $a \in [-1, +1]$ and $\gamma \in S$ then $a + \gamma \in S$ if $0 \leq a + \gamma \leq 1$

Similarly $a - \gamma \in S$ if $0 \leq a - \gamma \leq 1$.

(It should be noted that the validity of Theorem 4 does not depend on assumption (A4). It has been included to simplify life and to produce the result called Lemma 5 below).

To such a lattice S we shall attach on X a pseudo metric ρ_S or simply ρ by the prescription.

$$\rho(x, y) = \sup_{\gamma} \{|\gamma(x) - \gamma(y)|; \gamma \in S\}.$$

The norm $\|\gamma\|$ of a function will be the sup norm $\|\gamma\| = \sup_x |\gamma(x)|$.

Lemma 4 *Let S satisfy conditions (A1) to (A4) and let ρ be the pseudo metric S defined by S . Let A and B be two subsets of X such that for some $m \geq 1$*

$$\inf\{\rho(x, y); x \in A, y \in B\} > \frac{1}{m}.$$

Let \bar{S} be the closure of S in the space $\mathcal{F}(X, \mathbb{R})$ of numerical functions on X for the topology of uniform convergence on the ρ -precompact subsets of X .

Then there is an element $f \in m\bar{S}$ such that $0 \leq f \leq 1$, $f(x) = 0$ for $x \in A$ and $f(y) = 1$ for $y \in B$.

Proof. Let $\{x_1, x_2, \dots, x_n\}$ be a finite subset of A and let y be a specified element of B . Consider a pair (x_j, y) . By definition of ρ there is some $\gamma \in S$

such that $|\gamma(x_j) - \gamma(y)| > \frac{1}{m}$. One can assume $0 \leq \gamma(x_j) < \gamma(y) \leq 1$. If the reverse inequality held on would replace γ by $1 - \gamma$. Let $c_j = [\gamma(y) - \gamma(x_j)]^{-1}$. Define v_j by $c_j\{[(\gamma \vee a) \wedge b] - a\}$ where $a = \gamma(x_j)$ and $b = \gamma(y)$. This v_j is an element of mS . Now let $u = \min_j v_j$. Then $u \in mS$, $0 \leq u \leq 1$. Also $u(x_j) = 0$ for each x_j and $u(y) = 1$.

Fix the value of y and take the pointwise infimum of all such functions for all finite subsets $\{x_1, \dots, x_n\}$ of A . Let it be g . Then g vanishes on A and $g(y) = 1$. Thus g also belongs to $m\bar{S}$ since in mS pointwise convergence implies precompact convergence. This procedure can be repeated for all $y \in \mathcal{B}$. The pointwise supremum of all functions obtained in this way still belongs to $m\bar{S}$. It gives the desired function f . \square

The reader may have noticed the similarity of this proof with the usual proof of the Stone - Weierstrass theorem. The similarity will become even more apparent in our next result (which however will not be needed for our main arguments!)

Lemma 5 *Let S satisfy (A1) to (A4) and let ρ be the pseudo-metric it defines. Then \bar{S} consists exactly of those functions γ defined on X that satisfy $0 \leq \gamma \leq 1$ and $|\gamma(x) - \gamma(y)| \leq \rho(x, y)$ for all pairs (x, y) of elements of X .*

Proof. Taking equivalence classes, if necessary, one may assume that ρ is a metric. One can also complete X for this metric getting a completion \bar{X} where each precompact subset of X has a compact closure in \bar{X} . The function $\gamma \in \bar{S}$ extend by continuity to all of \bar{X} . They still satisfy the Lipschitz condition $|\gamma(x) - \gamma(y)| \leq \rho(x, y)$ for all pairs (x, y) of elements of \bar{X} .

Now the standard argument in the proof of the Stone - Weierstrass theorem says that if K is a compact subset of \bar{X} in order that a function f defined on K be approximable *uniformly on K* by elements of S it is necessary and sufficient that for every pair (x, y) of elements of K and every $\epsilon > 0$ there exist some element $\gamma_{x,y,\epsilon}$ of S such that $|f(x) - \gamma_{x,y,\epsilon}(x)| < \epsilon$ and $|f(y) - \gamma_{x,y,\epsilon}(y)| < \epsilon$.

The argument showing this is the same as the argument carried above for Lemma 1.

Now suppose that f satisfies $|f(x) - f(y)| \leq \rho(x, y)$ for all pairs (x, y) of elements of X and that $0 \leq \rho \leq 1$. In order to prove that for each triplet (x, y, ϵ) with x and y in $K \subset \bar{X}$ one can obtain the two-point approximation

described above, it is sufficient to show that it can be obtained for pairs (x, y) of elements of X . This is true of course only if one extends f to \bar{X} by continuity.

Now if x and y belong to X there is some $\gamma_1 \in S$ such that $|\gamma_1(x) - \gamma_1(y)| \geq \rho(x, y) - \frac{\epsilon}{4}$. Since f satisfies the desired Lipschitz condition and since S is convex, there is some $\gamma_2 \in S$ such that $||\gamma_2(x) - \gamma_2(y)| - |f(x) - f(y)|| < \frac{\epsilon}{2}$. Suppose for instance $f(x) = a < b = f(y)$, and replacing γ_2 by $1 - \gamma_2$ if necessary suppose $\gamma_2(x) \leq \gamma_2(y)$. One can also suppose that $0 \leq [f(y) - f(x)] - [\gamma_2(y) - \gamma_2(x)] < \frac{\epsilon}{4}$. Now let $\alpha = \gamma_2(x) \leq \gamma_2(y) = \beta$ and let $\gamma_3 = (\gamma_2 \vee \alpha) \wedge \beta - \alpha$. This is still in S . It is zero at x and equal to $\beta - \alpha$ at y . Also $0 \leq \gamma_3 \leq \beta - \alpha \leq f(y) - f(x)$. Let $\gamma_4 = a + \gamma_3$. Then $\gamma_4(x) = f(x)$ and $\gamma_4(y) = a + (\beta - \alpha) \leq f(y) \leq a + (\beta - \alpha) + \epsilon$. Also $0 \leq \gamma_4 \leq 1$ hence $\gamma_4 \in S$ completing the proof of the result \square

As already said above this lemma is not essential to the rest of our proof. It was included to give a clearer picture of what is happening. By contrast the following result will be very useful.

Lemma 6 *Let S satisfy (A1) to (A4). Let f be a real valued function defined on X and such that $0 \leq f \leq 1$. Assume also that for a particular pair (k, m) of integers the inequality $\rho(x, y) \leq \frac{1}{m}$ implies $|f(x) - f(y)| \leq \frac{1}{k}$. Then there is a $g \in m\bar{S}$ such that $0 \leq g \leq f \leq g + \frac{2}{k}$.*

Proof. Let $B_j = \{x : f(x) \leq j/k\}$ for $j = 1, 2, \dots, k$. For each $j < k$ consider the pair (B_j, B_{j+1}^c) . If $x \in B_j$ and $y \in B_{j+1}^c$ then $\rho(x, y) > \frac{1}{m}$ since $f(y) - f(x) > \frac{1}{k}$. Thus, by Lemma 1 there is a $u_j \in \bar{S}$ such that $0 \leq u_j \leq 1$ and such that $u_j(x) = 0$ for $x \in B_j$ and $u_j(y) = 1$ for $y \in B_{j+1}^c$. Let $u_0 = 0$ and let $g = \frac{1}{k} \sum_{j=0}^{k-1} u_j$. We claim that $0 \leq g \leq f \leq g + \frac{2}{k}$. To see this consider a particular point x element of a set $B_i^c \cap B_{i+1}$. For every $j \geq i + 1$ one has $u_j(x) = 0$. For $0 < j \leq i - 1$ one has $u_j(x) = 1$. Therefore if $i \leq k - 1$

$$\sum_{j=0}^{k-1} u_j(x) = \sum_{j=0}^i u_j(x) = (i - 1) + u_i(x),$$

so that $\frac{(i-1)}{k} \leq g(x) \leq \frac{i}{k} \leq f(x) \leq \frac{i+1}{k}$. This gives the desired result. \square

Consider now a real valued function φ defined on the set S . We shall call such a function *linear* if $\varphi[\alpha u + \beta v] = \alpha \varphi(u) + \beta \varphi(v)$ for all systems (α, β, u, v) such that u and v and $\alpha u + \beta v$ are in S . It will be called *positive*

if $u \in S$, $v \in S$ and $u \leq v$ implies $\varphi(u) \leq \varphi(v)$. Let L be the linear space spanned by \bar{S} . One can write L as $H - H$ where $H = \bigcup\{m\bar{S}; m \geq 1\}$ is the convex cone generated by \bar{S} .

Lemma 7 *Assume that S satisfies (A1) to (A4). Let φ be a linear functional that is defined and positive on \bar{S} . Then it possesses a unique positive linear extension to the space D_ρ of all bounded uniformly continuous functions for the pseudo-metric ρ attached to S .*

Proof. The functional φ extends to $H = \bigcup\{m\bar{S}; m \geq 1\}$ by writing $\varphi(f) = m\varphi(\frac{1}{m}f)$ if $f \in \bar{S}$. Because of the linearity assumption if $f \in n\bar{S}$ for $n > m$ then $m\varphi(\frac{1}{m}f) = n\varphi(\frac{1}{n}f)$ since $\frac{1}{n}f = \frac{m}{n}(\frac{1}{m}f)$ are all in \bar{S} . Thus the definition is consistent. Now suppose that $f \in D_\rho$ is such that $0 \leq f \leq 1$. According to Lemma 6 for any $\epsilon > 0$ there is a $g \in H$ such that $0 \leq g \leq f \leq g + \epsilon$. This implies that $\bar{\varphi}(f) = \inf\{\varphi(h); h \in H, h \geq f\}$ and $\underline{\varphi}(f) = \sup\{\varphi(g); g \in H, g \leq f\}$ are equal. One can easily check that the extension of φ to H satisfies our “linearity” requirement. So does the extension $\bar{\varphi}$ to positive elements of D_ρ . The extension to $L = H - H$ is immediate. Hence the result. \square

Lemma 7 does not say anything about the continuity of the extension of φ to D_ρ . This will be the subject of Theorem 4 below. Before we state the theorem, let us note the following.

Lemma 8 *Let S satisfy (A1) to (A4). Then a positive linear functional φ is \mathcal{U} -uniformly continuous on S if and only if it is \mathcal{U} -continuous at zero. The same applies to H , D_ρ^+ or D_ρ .*

Proof. The result would be an immediate consequence of known results in functional analysis if we had assumed that S was symmetric. We have not assumed that, but (A4) is strong enough to imply a sort of “symmetry”. Take a positive φ and two elements f and g of S . Then $\frac{1}{2}[f + (1 - g)] \in S$ and so does $[\frac{1}{2}(f - g) + \frac{1}{2}] \vee (\frac{1}{2})$ and $[\frac{1}{2}(f - g) + \frac{1}{2}] \wedge \frac{1}{2}$. Subtracting $1/2$ one sees that $\frac{1}{2}(f - g)^+$ and $\frac{1}{2}(f - g)^-$ both belong to S . Therefore if $0 \leq \gamma \leq \alpha$ on a precompact set K implies $\langle \varphi, \gamma \rangle < \epsilon/2$ for $\gamma \in S$, the inequality $|f - g| < \alpha/2$ on K will imply $|\langle \varphi, f - g \rangle| < \epsilon$ for all pairs (f, g) of elements of S . The same applies to the other spaces listed: H , D_ρ^+ etc. Hence the statement \square

Note also the following

Lemma 9 *Let φ be positive on D_ρ in order that φ be \mathcal{U} -continuous it is sufficient that for each $\epsilon > 0$ there be a precompact set K such that $\gamma \in D_\rho$, $0 \leq \gamma \leq 1$, $\gamma(x) = 0$ for $x \in K$ implies $\langle \varphi, \gamma \rangle < \epsilon$.*

Proof. Suppose $f \in D_\rho$ is such that $|f| < \epsilon$ on K . Then $(f \vee \epsilon) - \epsilon$ vanishes on K . However $\langle \varphi, f \rangle \leq \langle \varphi, (f \vee \epsilon) \rangle = \langle \varphi, (f \vee \epsilon) - \epsilon \rangle + \epsilon \langle \varphi, 1 \rangle$. Hence the conclusion. \square

Theorem 4 *Let S be a set that satisfies (A1) to (A4) on a set X . Let ρ be the pseudo-metric attached to S and let D_ρ be the corresponding set of bounded uniformly continuous functions on X . On D_ρ , or subsets of it, let \mathcal{U} be the structure of uniform convergence on the ρ -precompact subsets of X .*

Every positive linear functional defined on S and \mathcal{U} -continuous there admits a (unique) positive linear extension that is \mathcal{U} -continuous on the balls of D_ρ .

Furthermore, let Φ be a bounded set of positive linear functionals defined on D_ρ . The following conditions are all equivalent:

- a) The restriction of Φ to S is \mathcal{U} -equicontinuous at zero.*
- b) The restriction of Φ to each UEB subset of D_ρ is \mathcal{U} -equicontinuous*
- c) The restriction of Φ to a ball $\{\gamma : \gamma \in D_\rho; \|\gamma\| \leq 1\}$ of D_ρ is \mathcal{U} -equicontinuous.*

Remark. For statements (b) and (c) one should really say \mathcal{U} -uniformly equicontinuity. However, by Lemma 8, this is equivalent to equicontinuity at zero.

Proof. A φ that is \mathcal{U} -continuous at zero is already uniformly continuous on S . Thus it has an extension by continuity to \bar{S} . Lemma 7 says that this extension has a unique positive linear extension to D_ρ . Thus, below, we shall make no notational difference between φ defined on S or on the whole of D_ρ .

It is obvious that (c) \Rightarrow (b) \Rightarrow (a) and that this implies the continuity statement for an individual φ . Thus, it will sufficient to show that (a) \Rightarrow (c). To do this we shall use the following notation. For $u \in D_\rho$ the symbol $\varphi \cdot u$ will denote the functional defined by $\langle \varphi \cdot u, f \rangle = \langle \varphi, uf \rangle$ for all uf in the domain of φ .

If (a) holds on S then it also hold on \bar{S} for the continuous extensions of the elements of Φ . Thus we can assume Φ defined on \bar{S} and \mathcal{U} -equicontinuous.

Now choose an $\epsilon > 0$ such that $\epsilon \in (0, 1/2)$ and an integer m_1 such that $m_1\epsilon > 1$. Then there is a precompact set F_1 such that $f \in m_1\bar{S}$, $0 \leq f \leq 1$ and $f = 0$ on F_1 implies $\langle \varphi, f \rangle < \frac{\epsilon}{2}$ for all $\varphi \in \Phi$. Continuing sequentially, if $(m_1, F_1), (m_2, F_2) \dots (m_{n-1}, F_{n-1})$ have been selected, take $m_n > m_{n-1}$ so that $m_n\epsilon^n > 1$ and select a precompact set F_n so that $F_{n-1} \subset F_n$ and so that $f \in \bar{S}$, $0 \leq f \leq 1$, $f = 0$ on F_n implies $\langle \varphi, f \rangle < \epsilon^n/2$ for all $\varphi \in \Phi$.

For any set F let $F(\epsilon) = \{x : \rho(x, F) < \epsilon\}$. Let us consider the set, $B_1 = F_1$, $B_2 = \{F_2 \cap [B_1(\epsilon)]\} \cup B_1, \dots$ and so forth so that

$$B_{j+1} = B_j \cup [B_j(\epsilon^j) \cap F_{j+1}].$$

These are precompact sets and their union $K = \cup_j B_j$ is also precompact since every element of K is within $(1 - \epsilon)^{-1}\epsilon^j$ of the precompact set B_j . For each integer j let u_j be a function $u_j \in m_j\bar{S}$ such that $0 \leq u_j \leq 1$ and such that $u_j(x) = 1$ for $x \in B_j$ but $u_j(x) = 0$ for $x \in [B_j(\epsilon^j)]^c$. Let $v_j = u_1 \wedge u_2 \wedge \dots \wedge u_j$. Then $v_j \in m_j\bar{S}$. Consider the difference $v_j - v_{j+1} = v_j - (v_j \wedge u_{j+1})$. Since $u_{j+1}(x) = 1$ for $x \in B_{j+1}$ the difference $v_j - v_{j+1}$ vanishes on B_{j+1} . Also $v_j \leq u_j$ vanishes on $[B_j(\epsilon^j)]^c$. Thus $v_j - v_{j+1}$ vanishes on $B_{j+1} \cup [B_j(\epsilon^j)]^c$. Since B_{j+1} contains $B_j(\epsilon^j) \cap F_{j+1}$, this implies that $v_j - v_{j+1}$ vanishes on F_{j+1} . Both v_j and v_{j+1} belong to $m_{j+1}\bar{S}$. Thus arguing as in Lemma 8, one sees that

$$\langle \varphi, v_j - v_{j+1} \rangle < \epsilon^{j+1}.$$

Now consider the decreasing sequence $\varphi \geq \varphi \cdot v_1 \geq \varphi \cdot v_2 \geq \dots \geq \varphi \cdot v_j \geq \varphi \cdot v_{j+1} \geq \dots$. It has a limit in norm, say ψ , and this ψ is such that $\|\varphi - \psi\| \leq \sum \epsilon_j \leq (1 - \epsilon)^{-1}\epsilon$.

Let f , $0 \leq f \leq 1$ be an element of $H = \cup_m(m\bar{S})$ such that $f(x) = 0$ for $x \in K$. However f vanishes on $\mathcal{B}_n \subset K$. Thus, for $\varphi \in \Phi$ one has $\langle \psi, f \rangle < \epsilon^n$. Since n is arbitrary this implies $\langle \psi, f \rangle = 0$. This entails that $\langle \varphi, f \rangle = \langle \psi, f \rangle + \langle \varphi - \psi, f \rangle \leq 0 + \|\varphi - \psi\| < (1 - \epsilon)^{-1}\epsilon$. The desired result follows then from Lemma 8 and 9 at least for the balls of D_ρ^+ . However that is enough to imply \mathcal{U} -equicontinuity on the unit ball of D_ρ . This completes the proof. \square

Corollary 2 *If the structure \mathcal{V} of (X, \mathcal{V}) is metrisable then the spaces \mathcal{M}_p and \mathcal{M}_u of Section 3 are the same.*

Indeed, let $\rho, \rho \leq 1$ be a metric compatible with \mathcal{V} . The set S of positive functions f such that $0 \leq f \leq 1$ and $|f(x) - f(y)| \leq \rho(x, y)$ satisfies all the properties (A1) to (A4).

Corollary 3 *Let (X, \mathcal{V}) be a uniform space. The space \mathcal{M}_u is identical with the space of bounded functionals μ such that for every metric space Y and every uniformly continuous map f of X into Y the image $f\mu$ of μ belongs to \mathcal{M}_p of Y .*

Proof. One can decompose μ into its positive and negative parts. Thus it is enough to prove the result assuming that $\mu \geq 0$. By Theorem 4, $f\mu$ that belongs to \mathcal{M}_u of Y must also be in \mathcal{M}_p of Y . Conversely if S is a UEB of X , it defines a pseudo metric ρ . By passage to a quotient one obtains a metric space Y whose Lipschitz functions of coefficient unity reproduce on X a set that contains S . Thus the condition is certainly sufficient.

6. Relations with Radon measures.

We have already mentioned Radon measures on a *compact* set in Section 2. There are various extensions of the definition. For completely regular spaces and bounded measures, see Le Cam [1957]. For more general topological spaces and unbounded measures see Schwartz [1973]. The natural definition in the context of the present paper is that of a *tight* linear functional. A linear functional φ defined on a convex symmetric set of bounded numerical functions Γ on a topological space X is called tight on Γ if for every $\epsilon > 0$ there is a compact $K \subset X$ and a $\delta > 0$ such that $f \in \Gamma$, $|f| \leq 1$ and $|f(x)| < \delta$ for $x \in K$ implies $|\langle \varphi, f \rangle| < \epsilon$.

Note that this definition refers only to the set Γ and the compact subsets of X . It does not say anything about the domain on which φ might be defined or extended.

Now take a set X with a separated uniform structure \mathcal{V} and with set of bounded uniformly continuous functions D . A positive linear functional φ is tight on D for \mathcal{V} if it is continuous at zero on $\{\gamma : \gamma \in D, |\gamma| \leq 1\}$ for the structure of uniform convergence on *compact* subsets of X . If (X, \mathcal{V}) is complete this is the same as continuity for the structure \mathcal{U} of uniform convergence on the *precompact* subsets of (X, \mathcal{V}) .

If φ is positive tight on D for \mathcal{V} it admits an extension by the Bourbaki - MacShane procedure. One takes functions u that are pointwise supremum of subsets $S \subset D$ and write

$$\langle \varphi, u \rangle = \sup\{\langle \varphi, \gamma \rangle; \gamma \in D, \gamma \leq u\}.$$

This gives an extension to lower semicontinuous functions that are bounded from below. Similarly if v is upper semicontinuous bounded above, one writes $\langle \varphi, v \rangle = \inf\{\langle \varphi, \gamma \rangle; \gamma \in D, \gamma \geq v\}$. Finally if f is such that for every $\epsilon > 0$ there is a lower semicontinuous u (bounded from below) and an upper semicontinuous v (bounded from above) such that $v \leq f \leq u$ and $\varphi(u) - \varphi(v) < \epsilon$ one lets $\langle \varphi, f \rangle$ be the intersection of all the brackets $[\varphi(v), \varphi(u)]$ obtained in the procedure just described. This extends φ to a positive linear functional on a large space of numerical functions on X . We shall call it the Radon extension (or the Bourbaki extension) of the original φ . It is easily seen that the bounded functions in the Bourbaki-Radon extension of a φ tight on (D, \mathcal{V}) can also be obtained as follows. One completes X for the smallest uniform structure that makes the elements of D continuous and extend D to the compact completion \bar{X} where it becomes $C(\bar{X})$. Then φ is tight on (D, \mathcal{V}) if and only if X is the Bourbaki-Radon domain of φ extended from $C(\bar{X})$ and if $\varphi(\bar{X} \setminus X) = 0$. Thus making a Radon extension from (X, D) or from $(\bar{X}, C(\bar{X}))$ gives the same domain of extension as far as subsets of X are concerned. This can be summarized by saying that φ on (X, D) admits a Bourbaki-Radon extension if and only if it is tight on (X, D) for \mathcal{V} .

There are other functionals that admit extension by the Bourbaki-MacShane procedure. All the positive linear functionals that are τ -smooth on (X, D) admit such extensions, but they are not necessarily such that $\varphi(\bar{X} \setminus X) = 0$, just such that each compact subset of $\bar{X} \setminus X$ has measure zero. (See Le Cam 1957).

The word “tight” as originally used in Le Cam [1957] was meant to apply to a *set* of linear functionals. A set Φ of bounded linear functionals on D was called “tight” if it was uniformly bounded and uniformly continuous at zero on the unit ball of D for the structure of uniform convergence on the *compacts* of (X, \mathcal{V}) . Thus if X is complete for \mathcal{V} a bounded set of linear functionals is tight on D if and only if it is \mathcal{U} -equicontinuous on the unit ball of D . The word tight was applied to single linear functionals by abuse of language, regarding a single linear functional a the set that consists of that one functional.

Theorem 4 of Section 5 can also be stated in an equivalent form as follows.

Theorem 5 *Let (X, \mathcal{V}) be an arbitrary uniform space. Let Φ be a bounded set of positive linear functionals on D . Then Φ is equicontinuous for \mathcal{U} on the unit ball of D if and only if for every uniformly continuous map f of (X, \mathcal{V}) into a complete metric space (Y, ρ) the image of Φ by f is tight on (Y, ρ, D_ρ) .*

Another relation with Radon measures is an extension of Theorem 1, Section 2 as follows.

Theorem 6 *Let (X_1, \mathcal{V}_1) be a separated uniform space with a positive finite Radon measure μ . Let (X_2, \mathcal{V}_2) be another uniform space with space of bounded uniformly continuous functions D_2 . Let f be a map from X_1 to X_2 . Assume that*

- 1) *for every $\gamma \in D_2$ the composed map $\gamma \circ f$ is in the Bourbaki-Radon domain of μ .*
- 2) *The cardinal of the image $f(X_1)$ is not measurable (no two valued non-trivial probability measure).*

Then the image $f\mu$ of μ by f belongs to \mathcal{M}_u on (X_2, \mathcal{V}_2) .

Indeed all the uniformly continuous images of $f\mu$ in metric spaces must belong to $\mathcal{M}_u = \mathcal{M}_p$ there, by Theorem 1, Section 2.

It would be nice to have an extension of such a theorem to maps between linear functionals that are not necessarily obtained from a function f .

Suppose for instance that (X_i, \mathcal{V}_i) are two uniform spaces with respective sets of bounded uniformly continuous functions D_i . Consider a map A from D_2 to bounded integrable functions for a Radon measure μ on (X_1, \mathcal{V}_1) . Suppose A is positive, such that $A1 = 1$ and such that if a sequence $\{\gamma_n\}$, $\gamma_n \in D_2$ decreases pointwise to zero on X_2 the images $A\gamma_n$ do the same on X_1 .

One might expect that such a map would be given by a Markov kernel mapping X_1 into elements of \mathcal{M}_u of X_2 . However, even if one lives in a universe where there are no measurable cardinals (or strongly inaccessible ones) this might not be the case.

Suppose for instance that there is a non atomic probability measure π defined on all the subsets of a discrete set X_2 with the cardinality of the

continuum. Consider the map from X_1 to linear functional on D_2 that assigns to each $x \in X_1$ the same $\pi_x = \pi$ on D_2 . A $\gamma \in D_2$ will be transformed to the constant $\int \gamma d\pi_x$, element of D_1 . However $\pi = \pi_x$ is not in \mathcal{M}_u of X_2 , neither is $\int \pi_x \mu(dx)$ for any non zero positive Radon measure μ on X_1 .

However it is possible to obtain an extension of Theorem 1, Section 2, to certain Markov kernels. For instance one can prove the following.

Theorem 7 *Let (X, \mathcal{V}) be a separated uniform space with a positive finite Radon measure λ . Let (Y, \mathcal{V}_2) be another uniform space and let $x \rightsquigarrow \pi_x$ be a map from X to probability measures on Y that are elements of \mathcal{M}_u on (Y, \mathcal{V}_2) .*

Assume that for every $\gamma \in D(Y, \mathcal{V}_2)$ the image $x \rightsquigarrow \pi_x \gamma = \int \gamma(y) \pi_x(dy)$ is in the Radon-Bourbaki domain of λ .

Assume also that the cardinal of Y does not admit nontrivial two valued probability measures.

Then the image of λ defined by $\nu = \int \pi_x \lambda(dx)$ (that is $(\nu, \gamma) = \int [\int \gamma(y) \pi_x(dy)] \lambda(dx)$ for $\gamma \in D(Y, \mathcal{V}_2)$) is an element of \mathcal{M}_u on (Y, \mathcal{V}_2) .

Proof. According to Theorem 4, Section 5 it is enough to prove that for any uniformly continuous map g of (Y, \mathcal{V}_2) is a metric space the image of ν belongs to \mathcal{M}_u (or \mathcal{M}_p).

The images of the individual π_x by g are also in \mathcal{M}_u by assumption. Thus we are reduced to prove the theorem for the case where Y is a metric space, which can be assumed to be complete without loss of generality.

Now proceed as in Theorem 1, Section 2 using a well-ordered family of open sets $\{G_j; j \in J\}$ in Y , and the sets $A_j = G_j \setminus [\cup_{i < j} G_i]$. We shall assume $\{G_j; j \in J\}$ covers Y . Each π_x yields a measure μ_x on J by first extending π_x to its Radon extension on Y , say $\bar{\pi}_x$, and then letting $\mu_x(S) = \bar{\pi}_x[\cup_{j \in S} A_j]$.

Let us first show that each μ_x is carried by a countable subset of J . Since π_x is carried by a countable union of compacts of Y it is enough to prove that if a Radon π on Y is carried by a compact K then it is carried by a countable subset of the family $\{A_j; j \in J\}$. Consider also the increasing family $\{U_\alpha\}$ with $U_\alpha = \cup_j \{G_j; j < \alpha\}$. If α is a limit ordinal then $\pi(U_j); j < \alpha$ increases to the limit $\pi(U_\alpha)$. Thus only a countable number of the $A_j; j < \alpha$ can be such that $\pi(A_j) > 0$. Now K is contained in some U_β where β is either a limit ordinal or has the form $\beta = \alpha + n$ for some natural integer n and a

finite or limit ordinal α . In either case only a countable family of the A_j can have positive measure.

Passing to J this means that each one of our μ_x has a countable support, say S_x in the set J considered as a discrete set. On the discrete J all subsets S are such that $\mu_x(S)$ is in the Radon domain of λ . Consider $m = \int \mu_x \lambda(dx)$ on J . It may have atoms which must be points since J is not measurable. The atoms form a countable set A . One can write each μ_x as $\mu_x = \mu'_x + \nu_x$ where μ'_x is the part of μ_x carried by A and ν_x is carried by A^c . Then $m = m' + \nu$ where m' is on A and $\nu = \int \nu_x \lambda(dx)$ is on A^c and non atomic. It will be sufficient to show that $\nu = 0$. To do this we shall prove that if not zero, then, contrary to its definition, it must have some atomic part.

Removing a measurable subset of X if necessary one can assume $\nu_x(A^c) > 0$ for all $x \in X$. Then take a first x_1 such that ν_{x_1} is not zero. It has a finite or countable support S_1 in J . Let B_1 be the set of x is such that $\nu_x(S_1) > 0$. Proceeding along the ordinals, suppose that for each $\alpha < \beta$ one has selected an x_α with the support of ν_{x_α} equal to S_α and the corresponding set B_α of points x such that $\nu_x(S_\alpha) > 0$. Then let $T(\beta) = \cup_{\alpha < \beta} (S_\alpha; \alpha < \beta)$. Take a further x_β such that $\nu_{x_\beta}[T(\beta)] = 0$ but such that ν_{x_β} is not zero, if there is such point. Continue as before.

The process will stop at some ordinal (not larger than J in cardinality). Let Z be the segment of the ordinals so used. For any $x \in X$ let $z(x) = \alpha$ if $x \in B_\alpha$. This gives a map from x into Z . Indeed, suppose that $x \in X$ does not belong to any B_α . This means that $\nu_x(S_\alpha) = 0$ for all $\alpha \in Z$. Hence also $\nu_x(T_\alpha) = 0$ for all $\alpha \in Z$. This would allow the construction to be carried out further than Z . Therefore $\cup_{\alpha \in Z} B_\alpha = X$. Consider any subset, say W of Z . We claim that the inverse image $z^{-1}(W)$ is in the domain of λ . Indeed consider any particular $\alpha \in W$ the set B_α is the set of x 's such that $\nu_x(S_\alpha) > 0$. Thus the union $\cup_{\alpha \in W} B_\alpha$ is the set points x such that $\nu_x(S_\alpha) > 0$ for some $\alpha \in W$, that is the set of points x such that $\nu_x\{\cup_{\alpha \in W} S_\alpha\} > 0$ since the ν_x 's have countable support.

Now we have a map $X \rightsquigarrow Z$ by $x \rightsquigarrow z(x)$ that satisfies all the conditions of Theorem 1, Section 2 for Z considered as discrete. Therefore there exists a subset X_0 such that $\lambda(X \setminus X_0) = 0$ and such that $z(X_0)$ is separable, hence countable in the discrete Z .

Consider also sets $B = \cup_{\alpha \in z(X_0)} B_\alpha = z^{-1}[z(X_0)] \supset X_0$ and the set $T = \cup\{S_\alpha; \alpha \in z(X_0)\}$. For each $x \in B$, hence for each $x \in X_0$, the measure ν_x gives strictly positive mass to S_α . Therefore $\nu_x(T) > 0$ for all $x \in X_0$.

Since $\int_X \nu_\alpha \lambda(dx) = \int_{X_0} \nu_x \lambda(dx)$ we conclude that $\int \nu_x(T) \lambda(dx) > 0$ and that, since T is countable, $\int_X \nu_x \lambda(dx)$ must give positive mass to some point of T . This is contrary to the assumption that it has no atoms. Hence the result. The proof can now be completed exactly as in Theorem 1, Section 2, \square

One should note that here the fact that the measure λ of Theorem 7 is a Radon measure has been used. Just as in Theorem 1, Section 2, one could dispense with this assumption and assume only countable additivity of λ if the continuum \mathfrak{c} does not admit a non atomic probability measure. This is however a much stronger assumption than the non measurability of the cardinal of Y . Under this very weak assumption, the theorem might still hold for measures λ that are not Radon measures. We do not know of necessary and sufficient conditions on λ .

7. Continuous partitions of unity.

In this section we shall assume that X is a completely regular topological space and let $C^b(X)$ be the space of bounded continuous numerical functions on X . To link this set-up with the previous one, with a uniform structure \mathcal{V} and space of bounded uniformly continuous functions $D(X, \mathcal{V})$ one could use any one of the structures \mathcal{V} that makes all the elements of $C^b(X)$ uniformly continuous and is compatible with the topology of X . For reasons that will appear later we shall use the universal uniform structure $\bar{\mathcal{V}}$ defined by all the continuous pseudometrics on X . One concept that will play a particular role is that of a continuous partition of unity. This is defined as follows

Definition. *A continuous partition of unity (for X and $C^b(X)$) is a family $\{u_\alpha; \alpha \in A\}$ of elements of $C^b(X)$ subject to the conditions $0 \leq u_\alpha \leq 1$ and $\sum_\alpha u_\alpha(x) = 1$ for all $x \in X$. It is called locally finite if each $x \in X$ has a neighborhood that intersects only a finite number of the supports of the u_α .*

Let $\{u_\alpha; \alpha \in A\}$ be a partition of unity on X . Let $\mathcal{B} = \mathcal{B}(A)$ be the space of all bounded numerical functions on A . For each $\beta = (\alpha \rightsquigarrow \beta(\alpha))$ in \mathcal{B} let $T'\beta$ be the function defined on X by

$$(T'\beta)(x) = \sum_\alpha \beta(\alpha) u_\alpha(x).$$

Lemma 10 *For any partition of unity the map T' is a positive linear map from \mathcal{B} to $C^b(X)$. It transforms the unit ball $\{\beta : \beta \in \mathcal{B}, \|\beta\| \leq 1\}$ of \mathcal{B} into*

an equicontinuous subset B_u of $C^b(X)$. The transpose map T transforms the space \mathcal{M}_u built on (X, \bar{X}) into the space of bounded measures with countable support in the discrete space A .

Proof. Let x be an element of X . For any $\epsilon > 0$ there is a finite set F such that $\sum_{\alpha \in F} u_\alpha(x) \geq 1 - \epsilon/4$. Let $G = F^c$ and let $g(y) = \sum_{\alpha \in G} u_\alpha(y)$. Since $g = 1 - \sum_{\alpha \in F} u_\alpha$, it is continuous on X . Thus, there is a neighborhood V of x such that $g(y) < \epsilon/2$ for all $x \in V$. This implies

$$\left| \sum_{\alpha \in G} \beta(\alpha) u_\alpha(y) \right| \leq \|\beta\| \sum_{\alpha \in G} u_\alpha(y) < \epsilon/2,$$

for all $y \in V$. The sum over the finite set F of the type $\sum_{\alpha \in F} \beta(\alpha) u_\alpha(y)$, $\|\beta\| < 1$, are clearly equicontinuous. Hence the first statement.

Now, since we are using the universal structure $\bar{\mathcal{V}}$ of X , a bounded equicontinuous set B is also uniformly equicontinuous since $\sup\{|\gamma(x) - \gamma(y)|; \gamma \in B\}$ is a continuous pseudo-metric. Now note that for each fixed x the map $\beta \rightsquigarrow \sum_{\alpha} \beta(\alpha) u_\alpha(x)$ gives a measure with countable support on A . If $\mu \in \mathcal{M}_u(X, \bar{\mathcal{V}})$ then it is a limit uniformly on UEB sets of a bounded filter $\{\mu_\nu\}$ of measures with finite support.

Since $\int \gamma d\mu_\nu - \int \gamma d\mu$ converges to zero uniformly on the UEB set B_u their images $T\mu_\nu$ converge uniformly on the unit ball of \mathcal{B} . However this is equivalent to convergence in L_1 -norm. Hence $T\mu$, limit of $T\mu_\nu$ has countable support. Hence the result. \square

The reader should note that this result applies to $\mathcal{M}_u(X, \bar{\mathcal{V}})$ for the universal structure $\bar{\mathcal{V}}$, not necessarily to weaker structures \mathcal{V} on X . The structure $\bar{\mathcal{V}}$ may be remote with very few precompact sets and therefore a very small \mathcal{M}_u space. Think for instance of the set Q of rational numbers in $[0, 1]$. It is complete for a certain uniform structure \mathcal{V} compatible with its topology. Indeed any F_σ or intersection of F_σ in a complete space admits a uniform structure for which it is complete. Hence Q is also complete for its universal structure $\bar{\mathcal{V}}$. A set $S \subset Q \subset [0, 1]$ cannot be precompact if its closure in $[0, 1]$ contains any points not in Q . That is a $\bar{\mathcal{V}}$ precompact set $S \subset Q$ must be in the complement of an open neighborhood of the irrationals in $[0, 1]$. Since each compact subset of Q is also $\bar{\mathcal{V}}$ precompact, this is a necessary and sufficient condition. However the elements of $\mathcal{M}_u(Q, \bar{\mathcal{V}})$ are precisely the σ -additive measures carried by Q . This will follow from results given below but can be seen as follows. Call a linear functional μ on $C^b(Q)$ a σ -smooth

functional on Q if for every sequence $\{\gamma_n\}$ of $C^b(Q)$ that decreases pointwise to zero on Q one has $\langle \mu, \gamma_n \rangle \rightarrow 0$. Now let γ_n be such a sequence, with $\gamma_1 = 1$. It forms an equicontinuous set. Indeed to show this one can repeat the argument of Lemma 10: For a given x , there is a γ_n such that $\gamma_n(x) < \epsilon/4$ hence $\gamma_n(x) < \epsilon/2$ in some neighborhood of x . The $\gamma_0, \gamma_1, \dots, \gamma_{n-1}$ form an equicontinuous set. Thus for each $\epsilon > 0$ there is some neighborhood V of u such that $y \in V$ implies $|\gamma_n(y) - \gamma_n(x)| < \epsilon$ for all γ_n . Any element μ of $\mathcal{M}_u(Q, \bar{V})$ must be continuous on $\{\gamma_n; n = 0, 1, 2, \dots\}$ for the uniform convergence on compacts. However by Dini's theorem the γ_n tend to zero uniformly on compacts. Thus $\langle \mu, \gamma_n \rangle \rightarrow 0$ as $n \rightarrow \infty$. In other words the elements of $\mathcal{M}_u(Q, \bar{V})$ can be written in the form $\langle \mu, \gamma \rangle = \sum c_n \gamma(x_n)$ for some sequence $x_n \in Q$.

An interesting corollary of Lemma 10 is as follows.

Lemma 11 *Let μ be an element of $\mathcal{M}_u(X, \bar{V})$ and let $\gamma \in C^b(X)$. Then for every continuous partition of unity $\{u_\alpha; \alpha \in A\}$ the value of $\langle \mu, \gamma \rangle$ is the sum $\sum_{\alpha \in A} \langle \mu; \gamma u_\alpha \rangle$ limit of the finite sums $\sum_{\alpha \in F} \langle \mu, \gamma u_\alpha \rangle$ along the filter of finite subsets F of A .*

Proof. It is enough to prove this for $\mu \geq 0$ and $\gamma \geq 0$. Define another measure ν by $\langle \nu, f \rangle = \langle \mu, \gamma f \rangle$ if $f \in C^b(X)$. Construct the map T transpose of the map $\beta \rightsquigarrow \sum_{\alpha} \beta(\alpha) u_\alpha$ from $\mathcal{B}(A)$ to $C^b(X)$. Then let $\varphi = T\nu$. Here $\nu \in \mathcal{M}_u$, hence φ is carried by a countable subset of A . Thus for each $\epsilon > 0$ there is a finite set $F \subset A$ such that $\varphi(F^c) < \epsilon$ or equivalently

$$\sum_{\alpha \in F^c} \langle \nu, u_\alpha \rangle = \sum_{\alpha \in F^c} \langle \mu, \gamma u_\alpha \rangle < \epsilon.$$

The result follows. \square

To go further, recall that a linear functional μ on $C^b(X)$ is τ -smooth if for every decreasing directed family $\{f_\alpha\}$ of elements of $C^b(X)$ that decreases to zero pointwise on X the values $\langle \mu, f_\alpha \rangle$ tend to zero. It is known (see Le Cam 1957) that τ -smooth functionals form a band. In particular μ is τ -smooth if and only if μ^+ and μ^- are τ -smooth.

Lemma 12 *Assume that X is a paracompact space. A positive linear functional φ is τ -smooth on $C^b(X)$ if and only if it possesses the following property:*

(π) For every $\epsilon > 0$ and every locally finite partition of unity $\{u_\alpha; \alpha \in A\}$ there is a finite set $F \subset A$ such that $\langle \varphi, \sum_{\alpha \in F^c} u_\alpha \rangle < \epsilon$.

Proof. The property is clearly necessary since $\sum_{\alpha \in F^c} u_\alpha$ decreases to zero along the filter of finite subsets $F \subset A$.

Conversely let φ be a positive linear functional on $C^b(X)$. Assume $\langle \varphi, 1 \rangle = \|\varphi\| = 1$. Let A be a directed set and let $\{f_\alpha; \alpha \in A\}$ be a family such that $f_\alpha \in C^b(X)$, $0 \leq f_\alpha \leq 1$ and such that, along A , f_α decreases pointwise to zero on X . Take an $\epsilon \in (0, 1)$ and let $G_\alpha = \{x : f_\alpha < \epsilon\}$. This yields an open cover of X . Since X is paracompact there is a locally finite refinement say $\{G'_\tau; \tau \in T\}$ of $\{G_\alpha\}$ and a continuous partition of unity $\{u_\tau; \tau \in T\}$ such that u_τ has its support contained in G'_τ . It follows from property (π) that there is a finite set $F \subset T$ such that $\langle \mu, \sum_{\tau \in F^c} u_\tau \rangle < \epsilon$ hence $\langle \mu, f_\alpha \sum_{\tau \in F^c} u_\tau \rangle < \epsilon$ for all $\alpha \in A$. By construction each u_τ , $\tau \in F$ has its support in G'_τ contained in some $G_{\alpha(\tau)}$. Take then an $\alpha_0 \in A$ larger than all $\alpha(\tau)$, $\tau \in F$. For this α_0 one has $f_{\alpha_0}(x) < \epsilon$ for $x \in \cup\{G'_\tau; \tau \in F\}$ and for all $\alpha \geq \alpha_0$. Thus $\sum_{\tau \in F} \langle \mu, f_\alpha u_\tau \rangle < \epsilon$ for all $\alpha \geq \alpha_0$. This yields

$$\langle \mu, f_\alpha \rangle = \sum_{\tau \in F} \langle \mu, f_\alpha u_\tau \rangle + \langle \mu, f_\alpha \sum_{\tau \in F^c} u_\tau \rangle < 2\epsilon.$$

Hence the result. \square

Theorem 8 Let X be paracompact with universal uniform structure $\bar{\mathcal{V}}$. Let $\mathcal{M}_u = \mathcal{M}_u(X, \bar{\mathcal{V}})$ be the \mathcal{M}_u space of bounded linear functionals on $C^b(X)$ for the structure $\bar{\mathcal{V}}$. Let \mathcal{M}_τ be the space of bounded linear functionals that are τ -smooth on $C^b(X)$. Then $\mathcal{M}_u = \mathcal{M}_\tau$.

Proof. The combination of Lemmas 10 and 12 shows that $\mathcal{M}_u \subset \mathcal{M}_\tau$. To obtain the reverse implication use the corollary of Theorem 4, Section 5 and map X into a metric space Y by a continuous (hence uniformly continuous) map f . If $\mu \in \mathcal{M}_\tau$ on X then its image $\nu = f\mu$ on Y is also τ -smooth on Y . One can assume that Y is complete. Now on a complete metric space a τ -smooth ν is already in \mathcal{M}_p because it has a support, say S , that must be separable. \square

This leads to the following characterisation of the space $\mathcal{M}_u(X, \bar{\mathcal{V}})$ for an arbitrary completely regular space with *universal* uniform structure $\bar{\mathcal{V}}$.

Proposition 3 *Let X be an arbitrary completely regular space and let $\bar{\mathcal{V}}$ be its universal uniform structure. Let φ be a bounded linear functional on $C^b(X)$. The following conditions are all equivalent.*

- 1) $\varphi \in \mathcal{M}_u(X, \bar{X})$
- 2) *for every partition of unity $\{u_\alpha : \alpha \in A\}$, every $\gamma \in C^b(X)$ such that $|\gamma| \leq 1$ and every $\epsilon > 0$, there is finite set $F \subset A$ such that if $G = F^c$ then $|\langle \varphi, \gamma \sum_{\alpha \in G} u_\alpha \rangle| < \epsilon$.*
- 3) *Same as (2) but for locally finite partitions of unity.*
- 4) *If f is a continuous map of X into a paracompact space Y then the image $f\varphi$ of φ is τ -smooth on $C^b(Y)$.*
- 5) *Same as (4) but with Y metric instead of paracompact*

Proof. It follows from Lemma 10 and Theorem 8 that (1) implies all the other conditions. Also, clearly, (2) \Rightarrow (3) \Rightarrow (4) since any locally finite partition of unity $\{v_\alpha; \alpha \in A\}$ on Y yields a corresponding locally finite partition $\{v_\alpha \circ f; \alpha \in A\}$ on X . Thus it will be sufficient to show that (5) \Rightarrow (1). However this follows from Theorem 4, Section 5. \square

8. A complement to Section 2.

Theorem 1 of Section 2 involves maps of Radon measures into metric spaces. Theorem 6 of Section 6 involves Markov kernels and images of Radon measures. We shall now show that a supplementary result can be obtained for maps of Radon measures into paracompact spaces.

Theorem 9 *Let X be a uniform space and let λ be a positive finite Radon measure on X . Let $x \rightsquigarrow \pi_x$ be a map defined on X to probability measures on a paracompact space Y . Assume*

- 1) *the cardinal of Y is not two valued measurable*
- 2) *for each bounded continuous function γ on Y the functions $\int \gamma(y)\pi_x(dy)$ is in the domain of λ .*

3) each π_x is τ -smooth on $(Y, C^b(Y))$.

Then there is a closed Lindelöf subset $B \subset Y$ such that $\int \pi_x(B^c)\lambda(dx) = 0$.

Note. We have used π_x as a measure on subsets of Y . This is by usual abuse of language. If π is τ -smooth on $C^b(Y)$ then it possesses an extension by the MacShane - Bourbaki procedure to lower semicontinuous functions that are bounded from below and to upper semicontinuous functions bounded from above. Note that this extension is such that if $\{G_\alpha; \alpha \in A\}$ is an increasingly directed family of open sets then $\pi[\cup_\alpha G_\alpha]$ is the limit $\lim_\alpha \pi(G_\alpha)$.

Proof. According to Theorem 7, Section 6 the integral $\nu = \int \pi_x \lambda(dx)$ belongs to \mathcal{M}_u of Y for the universal structure $\bar{\mathcal{V}}$ of Y . Hence it is in \mathcal{M}_τ of $(Y, C^b(Y))$ and extends as indicated. Now ν has a support $B \subset Y$, the smallest closed set such that $\nu(B) = \nu(Y)$. This is because of the continuity relation just recalled for open sets. A union of open sets of measure zero for ν has measure zero for ν .

The set B , closed subset of a paracompact space is also paracompact. Let us show that it also has the Lindelöf property that every cover $\{G_\alpha; \alpha \in A_1\}$ of B by open sets has a *countable* subcover.

Indeed let $\{G_\beta\}$ be such a cover of B and let $\{u_\alpha; \alpha \in A_2\}$ be a continuous partition of unity such that each u_α has its support contained in some G_β and such that u_α is not identically zero. Then $\|\nu\| = \sum_\alpha \langle \nu, u_\alpha \rangle$. However that sum can have only a countable number of non zero terms. Since each u_α is non zero at some point of B it is strictly positive in some neighborhood of that point. Thus $\langle \nu, u_\alpha \rangle$ is strictly positive and the partition $\{u_\alpha; \alpha \in A_2\}$ is a countable partition. Taking for each u_α a G_β that contain the support of u_α gives the desired countable family.

9. Compactness criteria.

The best known compactness criteria for bounded measures on topological spaces are those derived from Prohorov's theorem: On a complete separable metric space, a set of probability measures is vaguely relatively compact if and only if it is tight.

The aim of the present section is to give some other criteria that look very much weaker than tightness and might conceivably be easier to verify.

We start with a uniform space (X, \mathcal{V}) as in Section 3. The space of bounded uniformly continuous numerical functions on (X, \mathcal{V}) will be called D as before and we shall describe criteria for relative compactness in the space \mathcal{M}_u on (X, \mathcal{V}) .

Note first the following:

Lemma 13 *Let S be a bounded subset of \mathcal{M}_u . It is relatively compact in \mathcal{M}_u for the induced structure $[\mathcal{V}]$ if and only if its restriction to each UEB subset of D is equicontinuous there.*

Proof. It is sufficient to consider UEB sets B that are compact for the structure \mathcal{U} of uniform convergence on precompact sets of X .

The result is then an easy consequence of Ascoli's theorem. \square

Very often one wants criteria of compactness for the weak topology $w(\mathcal{M}_u, D)$ instead of the stronger $[\mathcal{V}]$. If one looks for subsets of \mathcal{M}_u^+ then $w(\mathcal{M}_u, D)$ compactness and $[\mathcal{V}]$ -compactness mean the same thing since on \mathcal{M}_u^+ the weak topology coincides with the topology induced by $[\mathcal{V}]$. See Proposition 2, Section 4. Thus, for the case of positive measures, the compactness requirement for $[\mathcal{V}]$ is the same as a $w(\mathcal{M}_u, D)$ compactness requirement. For general bounded sets in \mathcal{M}_u compactness for $[\mathcal{V}]$ appears to be stronger than $w(\mathcal{M}_u, D)$ compactness. Yet, the condition of Lemma 13, equicontinuity on each UEB set is weaker than the usual "tightness" requirement of equicontinuity for \mathcal{U} on the unit ball of D .

It is a strange affair that for metrizable (X, \mathcal{V}) and for sets in \mathcal{M}_u^+ equicontinuity on a suitably rich sublattice UEB of D already implies equicontinuity for \mathcal{U} on the unit ball of D . See Theorem 4, Section 5.

Using a theorem of Grothendieck (1952) one can even get further criteria that look even weaker than equicontinuity for \mathcal{U} on each UEB. Let us restate Théorème 7 of Grothendieck (1952) (page 183) and show that it applies here

Théorème 7 (Grothendieck) *Let \mathcal{E} and \mathcal{F} be two separated (real) locally convex topological linear spaces in duality. Let $\{K_\alpha\}$ be a family of subsets of \mathcal{E} such that the closed convex symmetric hulls \tilde{K}_α of each K_α is $w(\mathcal{E}, \mathcal{F})$ compact. Assume that the \tilde{K}_α generates algebraically all of \mathcal{E} .*

Let A be a bounded subset of \mathcal{F} . Assume that the closed convex hull of A is complete for the structure of uniform convergence on the K_α .

- a) If the K_α are $w(\mathcal{E}, \mathcal{F})$ compact in order that A be $w(\mathcal{F}, \mathcal{E})$ relatively compact in \mathcal{F} it is necessary and sufficient that for each α the set of continuous functions defined by A on K_α be relatively compact in $C(K_\alpha)$ for pointwise convergence.
- b) Whether or not the K_α are closed, a necessary and sufficient condition for the $w(\mathcal{F}, \mathcal{E})$ relative compactness of A in \mathcal{F} is that for every sequence $\{y_n\}$, $y_n \in A$ and every sequence $\{x_k\}$ contained in some K_α the existence of the iterated limits

$$\lim_k \lim_n \langle x_k, y_n \rangle \quad \text{and} \quad \lim_n \lim_k \langle x_k, y_n \rangle$$

implies their equality.

Note. The word “bounded” used here means that for every neighborhood V of zero there is some finite number a such that $A \subset aV$. For the iterated limits $\lim_k \lim_n \langle x_k, y_n \rangle$ it is meant that for each k the limit $\ell_k = \lim_n \langle x_k, y_n \rangle$ exists and then that $\lim_k \ell_k$ exists. Similarly reversing the order of k and n .

Grothendieck’s result is obtainable as a result of a sequence of arguments that start with the basic result that if X is countably compact and Y completely regular then in the space $C(X, Y)$ of continuous functions from X to Y , with pointwise convergence, relative countable compactness and relative compactness are equivalent and equivalent to a modified version of the iterated limit condition: In the notation used here there is some point a such that each neighborhood of a encounters an infinity of lines and an infinity of columns of the matrix $\langle x_k, y_n \rangle$.

Now what does that have to say for our spaces D and \mathcal{M}_u ? Let $\{K_\alpha\}$ be the family of all convex symmetric UEB subsets of D that are closed for \mathcal{U} -convergence. It does generate D algebraically. In fact $D = \cup_\alpha K_\alpha$.

According to Theorem 2, Section 4 any ball $\{\mu; \|\mu\| \leq b\}$ of \mathcal{M}_u is complete for the uniform convergence on the K_α . Thus we are definitely in a situation where Théorème 7 is applicable to bounded (in norm) subsets of \mathcal{M}_u .

What it gives here is the following

Theorem 10 *Let D and \mathcal{M}_u be as usual on (X, \mathcal{V}) . Let S be a bounded (in norm) subset of \mathcal{M}_u . Then the following conditions are equivalent.*

- 1) S is relatively compact for $w(\mathcal{M}_u, D)$ in \mathcal{M}_u .

- 2) For each \mathcal{V} -compact UEB, K , of D the restrictions of S to K are pointwise relatively compact in $C(K)$.
- 3) For each sequence $\{\gamma_n\}$ contained in a UEB and each sequence $\{\mu_k\}$ contained in S the existence of the iterated limits

$$\lim_k \lim_n \langle \mu_k, \gamma_n \rangle \quad \text{and} \quad \lim_n \lim_k \langle \mu_k, \gamma_n \rangle$$

implies their equality.

The theorem is an immediate application of Théorème 7, as explained above. Note however the following facts. Lemma 13 requires the U -equicontinuity of S on each UEB. Condition (2) of Theorem 10 just requires that, on a set K that is a UEB, limit *pointwise* on K of elements of S be continuous on K .

The weakest requirement seems to be (3) of Theorem 10. It involves only *sequences* $\{\gamma_n\}$ contained in some UEB. There is a tremendous distance between such a condition and the equicontinuity for \mathcal{U} on the entire unit ball $\{\gamma \mid \gamma \in D, \|\gamma\| \leq 1\}$ of D that was our conclusion for Theorem 4, Section 4. Note however that this was only for \mathcal{M}_u^+ on a *metrisable* (X, \mathcal{V}) . Still, there should be possibilities to use the much weaker sounding criterion (3) of Theorem 10.

Another facet of Theorem 10 is that $w(\mathcal{M}_u, D)$ relative compactness is equivalent to equicontinuity on each UEB. There are results of D. Preiss [1973] that say that on the set of rationals Q of $[0, 1]$ a set S can be relatively compact in the set of positive Radon measures and still not equicontinuous on the unit ball of $C^b(Q)$ for uniform convergence on compacts. At first glance this may seem contrary to the conjunction of (1) of Theorem 10 above and of Theorem 4 Section 4. However, this is not the case. If one gives Q its universal uniform structure $\bar{\mathcal{V}}$ then $D(Q, \bar{\mathcal{V}})$ is $C^b(Q)$ and \mathcal{U} is the structure of uniform convergence on compact subsets of Q . Thus a set S of positive Radon measures on Q (arising from a subset of $\mathcal{M}_u^+(Q, \bar{\mathcal{V}})$) that is relatively compact for $w[\mathcal{M}_u, C^b(Q)]$ must be \mathcal{U} -equicontinuous on each UEB subset of $C^b(Q)$, that is, on every bounded equicontinuous subset of $C^b(Q)$. However the universal structure $\bar{\mathcal{V}}$ of Q is not metrisable and Theorem 4, Section 4, does not apply.

A theorem that may be applicable to such a case is the following

Theorem 11 *Let X be paracompact and let $D = C^b(X)$ for any norm bounded subset $S \subset \mathcal{M}_\tau$ the following conditions are equivalent*

- 1) S is $w(\mathcal{M}_\tau, D)$ countably relatively compact in \mathcal{M}_τ
- 2) S is $w(\mathcal{M}_\tau, D)$ relatively compact in \mathcal{M}_τ
- 3) The sets $S^+ = \{\mu^+; \mu \in S\}$ and $S^- = \{\mu^-; \mu \in S\}$ are relatively compact in \mathcal{M}_τ
- 4) Let $\{f_\alpha; \alpha \in A\}$ be a decreasingly directed family $f_\alpha \in D^+$ that decreases to zero pointwise on X . Then

$$\limsup_{\alpha} \sup_{\mu \in S} |\langle \mu, f_\alpha \rangle| = 0.$$

We shall not prove this here. It follows by a combination of the arguments used earlier in this paper. See also Granirer [1967]. However the words “countably relatively compact” used above should be clarified. What they mean is this: If F is a countable infinite subset of S , there is some $\mu \in \mathcal{M}_\tau$ such that each neighborhood of μ contains an infinite subset of F .

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