(Draft)

A desymmetrization lemma

by R. M. Dudley, L. LeCam, and A. de Araujo

Our purpose is to perfect a central limit theorem in Banach spaces due to N. Jain and M. Marcus by eliminating a symmetry hypothesis.

Let S be a compact metric space, $C(S)$ the space of continuous real-valued functions on S with sup norm. Let $X_1, X_2, \ldots,$ be independent, identically distributed $C(S)$-valued random variables with $EX_1(t) = 0$ and $EX_1(t)^2 < \infty$ for all $t \in S$. Let $T_n = (X_1 + \ldots + X_n)/n^{\frac{1}{2}}$. Let v-convergence, written $\xrightarrow[v]{}$, be the usual "weak" convergence of probability measures. Let $\mathcal{L}(X)$ denote the probability law (distribution) of any random variable X.

Let G be the Gaussian stochastic process on S with mean 0 and $EG(s)G(t) = EX_1(s)X_1(t)$ for all s and t, so that for any finite set $F \subset S$, by the finite-dimensional central limit theorem the laws of T_n restricted to F converge to the law of G restricted to F.

Let $X_1, X_1', X_2, X_2', \ldots,$ be i.i.d., and let T_n^s be the symmetrized variable defined by

$$T_n^s = n^{-\frac{1}{2}}\sum_{j=1}^{n}(X_j - X_j').$$

Then on any finite set F, the laws of T_n^s converge to that of $2^{\frac{1}{2}}G$, and $\mathcal{L}(T_n^s) = \mathcal{L}(-T_n^s)$.

<u>Lemma</u>. If $\mathcal{L}(T_n^s) \xrightarrow[v]{} \mathcal{L}(2^{\frac{1}{2}}G)$, then $\mathcal{L}(T_n) \xrightarrow[v]{} \mathcal{L}(G)$.

<u>Proof</u>. By assumption, we can choose a version of G which is a $C(S)$-valued random variable.

Since $\mathcal{L}(T_n^s)$ are uniformly tight, there are some f_n in $C(S)$ such that $\mathcal{L}(T_n - f_n)$ are uniformly tight and hence v-relatively compact (Parthasarathy [2], III.2.2 p. 59).

Let d be a metric metrizing v-convergence of laws on $C(S)$, e.g. Prokhorov's metric or the metric β defined below. Let

$$a_n = \inf_{f \in C(S)} d(\mathcal{L}(T_n - f), \mathcal{L}(G)).$$

If $a_n \not\to 0$, let $a_{n(k)} \geq \varepsilon > 0$, $n(k) \longrightarrow \infty$. By tightness there is a v-convergent subsequence $\mathcal{L}(T_{n(k(j))} - f_{n(k(j))})$. By the finite-dimensional central limit theorem, $f_{n(k(j))}$ converge pointwise to some function h and

$$\mathcal{L}(T_{n(k(j))} - f_{n(k(j))}) \underset{v}{\longrightarrow} \mathcal{L}(G - h).$$

It follows that h is continuous and

$$\mathcal{L}(T_{n(k(j))} + h - f_{n(k(j))}) \underset{v}{\longrightarrow} \mathcal{L}(G),$$

so $a_{n(k(j))} \longrightarrow 0$, a contradiction. Hence $a_n \longrightarrow 0$.

Thus we can choose $f_n \in C(S)$ such that $\mathcal{L}(T_n - f_n) \underset{v}{\longrightarrow} \mathcal{L}(G)$ and hence $f_n \longrightarrow 0$ pointwise. It remains to show that $f_n \longrightarrow 0$ uniformly.

The reason is that convergence of laws together with uniform boundedness of second moments implies convergence of means, uniformly in t. Details follow.

The f_n are uniformly bounded by the Banach-Steinhaus theorem, say

(1) $\quad |f_n(t)| \leq K < \infty$ for all n and t.

We recall that v-convergence is metrized by the dual-bounded-Lipschitz metric β defined as follows. For any separable metric space (X,d) (in this case $X = C(S)$ with sup-norm metric), and a real-valued function g on X, let

$$||g||_{BL} = \max(\sup_{x \in X} |g(x)|, \sup_{x \neq y} |g(x) - g(y)|/d(x,y)).$$

Let, for any probability measures μ and ν on X,

(2) $\quad \beta(\mu, \nu) = \sup\{|\int g d(\mu - \nu)| : \ ||g||_{BL} \leq 1\}.$

Then β metrizes v-convergence (R. Ranga Rao [3], and [1], Theorem 12 p. 262).

Since G is sample-continuous and Gaussian, it must be continuous in quadratic mean, and S is compact, so G has bounded covariance. It follows from (1) that

$$(3) \quad E(T_n - f_n)(t)^2 \leq 2K^2 + 2ET_n(t)^2 = 2K^2 + 2EG(t)^2 \leq C$$

for some constant $C < \infty$.

For any $M > 0$ and real number x let

$$x_M = \max(-M, \min(x, M)).$$

Then for any probability measures μ and ν on R,

$$(4) \quad \left|\int x\,d(\mu - \nu)\right| \leq \left|\int x_M\,d(\mu - \nu)\right| + \int |x - x_M|\,d(\mu + \nu)$$
$$\leq \left|\int x_M\,d(\mu - \nu)\right| + \int x^2\,d(\mu + \nu)/M.$$

We apply (4) with $\mu = \mu_n = \mathcal{L}((T_n - f_n)(t))$, $\nu = \mathcal{L}(G(t))$. By (2) and (3) we get

$$|f_n(t)| = |E(T_n - f_n)(t)| \leq M\beta(\mu_n, \nu) + 2C/M.$$

For any $\varepsilon > 0$, we can choose M large enough so that we have for n large enough $|f_n(t)| < \varepsilon$ for all t. Hence $\mathcal{L}(T_n) \xrightarrow[v]{} \mathcal{L}(G)$, Q.E.D.

Note. Since any separable Banach space is linearly isometric to a subspace of some $C(S)$, the Lemma carries over to all such spaces.

REFERENCES

1. R. M. Dudley, "Convergence of Baire measures," Studia Math. 28 (1966) 251-268.
2. K. R. Parthasarathy, Probability Measures on Metric Spaces (Academic Press, New York, 1967)
3. R. Rango Rao, "Some theorems on weak convergence of measures and applications," Ann. Math. Statist. 33 (1962) 659-680.