EXAMPLE **A** Let $X_1, \ldots, X_n$ be a random sample from a normal distribution having known variance σ^2. Consider two simple hypotheses:

$$H_0\colon \mu = \mu_0$$
$$H_A\colon \mu = \mu_1$$

where μ_0 and μ_1 are given constants. Let the significance level α be prescribed. The Neyman-Pearson Lemma states that among all tests with significance level α, the test that rejects for small values of the likelihood ratio is most powerful. We thus calculate the likelihood ratio, which is

$$\frac{f_0(\mathbf{X})}{f_1(\mathbf{X})} = \frac{\exp\left[\dfrac{-1}{2\sigma^2}\displaystyle\sum_{i=1}^{n}(X_i - \mu_0)^2\right]}{\exp\left[\dfrac{-1}{2\sigma^2}\displaystyle\sum_{i=1}^{n}(X_i - \mu_1)^2\right]}$$

since the multipliers of the exponentials cancel. Small values of this statistic correspond to small values of $\sum_{i=1}^{n}(X_i - \mu_1)^2 - \sum_{i=1}^{n}(X_i - \mu_0)^2$. Expanding the squares, we see that the latter expression reduces to

$$2n\overline{X}(\mu_0 - \mu_1) + n\mu_1^2 - n\mu_0^2$$

Now, if $\mu_0 - \mu_1 > 0$, the likelihood ratio is small if $\overline{X}$ is small; if $\mu_0 - \mu_1 < 0$, the likelihood ratio is small if $\overline{X}$ is large. To be concrete, let us assume the latter case. We then know that the likelihood ratio is a function of $\overline{X}$ and is small when $\overline{X}$ is large. The Neyman-Pearson lemma thus tells us that the most powerful test rejects for $\overline{X} > x_0$ for some x_0, and we choose x_0 so as to give the test the desired level α. That is, x_0 is chosen so that $P(\overline{X} > x_0) = \alpha$ if H_0 is true. Under H_0 in this example, the null distribution of $\overline{X}$ is a normal distribution with mean μ_0 and variance σ^2/n, so x_0 can be chosen from tables of the standard normal distribution. Since

$$P(\overline{X} > x_0) = P\left(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} > \frac{x_0 - \mu_0}{\sigma/\sqrt{n}}\right)$$

we can solve

$$\frac{x_0 - \mu_0}{\sigma/\sqrt{n}} = z(\alpha)$$

for x_0 in order to find the rejection region for a level α test. Here, as usual, $z(\alpha)$ denotes the upper α point of the standard normal distribution; that is, if Z is a standard normal random variable, $P(Z > z(\alpha)) = \alpha$. ■

This example is typical of the way that the Neyman-Pearson Lemma is used. We write down the likelihood ratio and observe that small values of it correspond in a one-to-one manner with extreme values of a test statistic, in this case $\overline{X}$. Knowing the null distribution of the test statistic makes it possible to choose a critical level that produces a desired significance level α.