Stat205B: Probability Theory (Spring 2003) **Lecture: 3**

Recurrence and Transience

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Reference: Section 5.3 of Durrett. Note the following consequence of the Strong Markov Property: if T is a stopping time for a Markov Chain (X_n) such that with probability one either $T = \infty$ or ($T < \infty$ and $X_T = i$), then given $T < \infty$, the process $(X_T, X_{T+1}, \dots)$ is a copy of the original chain with distribution $\mathbb{P}_i$, independent of $(X_0, \dots, X_T)$.

`Notations`

- $T_i = \inf\{n > 0, X_n = i\}$ = first hit of i.
- $T_i^k = \inf\{n | n > T_i^{k-1}, X_n = i\}$ = k-th hit of i.
- $\rho_{ji} = \mathbb{P}_j(Ti < \infty)$.
- $N_i = \sum_{n=0}^{\infty} 1(X_n = i)$ = total number of visits to i.

Theorem 1.1 $\mathbb{P}_j(T_i^k < \infty) = \rho_{ji}\rho_{ii}^{k-1}$

Definition 1.1 (Recurrent states, Transient states)

- *The state i is recurrent if $\rho_{ii} = 1$.*
- *The state i is transient if $\rho_{ii} < 1$.*

Proposition 1.2

- *State i is recurrent iff $\mathbb{P}_i(N_i = \infty) = 1$.*
- *State j is transient iff $\mathbb{P}_i(N_i = \infty) = 0$.*

`Notice`

$$\begin{aligned} \mathbb{E}_i(N_i) &= \mathbb{E}_i(\textstyle\sum_{n=0}^{\infty} 1_{(X_n=i)}) \\ &= \textstyle\sum_{n=0}^{\infty} \mathbb{E}_i(1_{(X_n=i)}) \\ &= \textstyle\sum_{n=0}^{\infty} \mathbb{P}_i(1_{(X_n=i)}) \\ &= \textstyle\sum_{n=0}^{\infty} P_{ii}^n \end{aligned}$$

This sum is infinite(finite) iff state i is recurrent (transient).

Definition 1.2 *Say state i leads to state j ($i \rightsquigarrow j$) if there exists a $n \geq 0$ such that $P_{ij}^n > 0$.*

Theorem 1.3 *If $i \rightsquigarrow j$ and i is recurrent then j is recurrent.*

Definition 1.3 (Irreducible Markov Chain) *The matrix P (or the Markov Chain) is called irreducible if for all states i,j we have $i \rightsquigarrow j$ (all states communicate).*

Proposition 1.4 *If P is irreducible then*

- *either all states are recurrent and for all states ij, $\mathbb{P}_i(N_j = \infty) = 1$.*
- *or all states are transient, and for all states ij, $\mathbb{P}_i(N_j = \infty) = 0$.*

Exercise 1.1 *Let $X_n = \sum_{k=0}^n \xi_k$ be a random walk in $\mathbb{Z}$ with $\mathbb{P}(\xi_n = 1) = p$ and $\mathbb{P}(\xi_n = -1) = q$.There are two cases,*

- *either $p = q = \frac{1}{2}$, then all states are recurrent.*
- *or $p \neq q$, then all states are transient.*

Exercise 1.2 *A nearest neighbour randon walk in $\mathbb{Z}^d$ is recurrent for $d \in \{1, 2\}$ and transient for $d \geq 3$.*

Exercise 1.3 *A Random Walk on an infinite-binary-tree is transient. (Consider the Markov Chain $D_n = distance(X_n, X_0)$).*

Our next topic is to look at the behavior of the distribution of X_n for large n. More precisely, does the limit $\lim_{n\infty} \mathbb{P}^n_{i,j}$ exist?

A first easy case to discuss is when state j is recurrent, then $\mathbb{P}^n_{i,j} \rightarrow_{n\infty} 0$ (since $\sum_{n=0}^{\infty} P^n_{j,i} < \infty$).

If j is recurrent, then look at $\mathcal{C} = \{i | j \rightsquigarrow i\}$. This $\mathcal{C}$ is a communication class and satisfies $\forall i \in \mathcal{C}, \sum_{j \in \mathcal{C}} P_{i,j} = 1$. So $(P_{i,j})_{i,j \in \mathcal{C}}$ is a new transition matrix, that we can study.

This discussion shows that we can consider only the irreducible case without loss of generality.