Stat205B: Probability Theory (Spring 2003) **Lecture: 23**

Local Martingales and Quadratic Variation

Lecturer: Matthieu Cornec *Scribe: Brian Milch* `milch@cs.berkeley.edu`

This lecture covers some of the technical background for the theory of stochastic integration. First, some notation: $M = (M_t)_{t \geq 0}$ is a process, and $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration. We assume $\mathcal{F}$ is right-continuous and complete ($\mathcal{F}_t$ includes the null sets for each t). If τ is a stopping time, then M^τ is M stopped at time τ:

$$M^\tau = (M_{t \wedge \tau})_{t \geq 0}$$

23.1 Local martingales

Definition 23.1 *A process M is a* **local martingale** *w.r.t. $\mathcal{F}$ if:*

1. *M is adapted to $\mathcal{F}$, that is, $\forall t \; M_t \in \mathcal{F}_t$*
2. *there exists a sequence (τ_n) of stopping times such that $\tau_n \uparrow \infty$ a.s., and M^{τ_n} is a true martingale for each n.*

Definition 23.2 *M is a* **local L^2 martingale** *if it satisfies Def. 23.1 with M^{τ_n} being an L^2 martingale for each n.*

Other terms of the form "local <*adjective*> martingale (e.g., local bounded martingale) are defined similarly: we require that each M^{τ_n} be an <*adjective*> martingale. Note that "<*adjective*> local martingale means something different: if we say that M is a bounded local martingale, we are saying that M is bounded and its a local martingale; were not saying anything special about the M^{τ_n}.

Remark 1: If M is a continuous local martingale, then we can take the M^{τ_n} to be bounded martingales. We can do this by letting $\tau_n = \inf\{t : |M_t| \geq n\}$; then since the paths are continuous, $|M^{\tau_n}| \leq n$.

Remark 2: Any continuous bounded local martingale is a true martingale. To see this, note that $M^{\tau_n} \uparrow M$, and since M is bounded we can apply the dominated convergence theorem.

Definition 23.3 *Define the variation of M over the interval $[0, t]$ as:*

$$V_t(\omega) = \sup_{\substack{n \in \mathbb{N} \\ 0 = t_0 < \cdots < t_n = t}} \sum_{i=1}^{n} |M_{t_i}(\omega) - M_{t_{i-1}}(\omega)|$$

Then M has **locally finite variation** *if $\forall t \; \exists C_t < \infty \; V_t < C_t$ everywhere.*

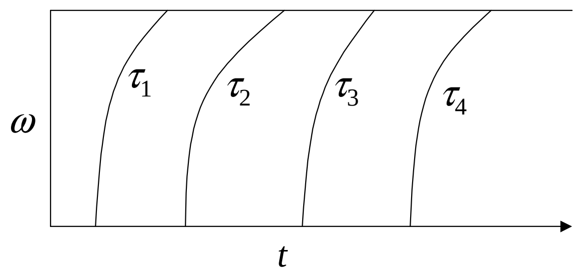

Figure 23.1: A localization argument involves proving a claim about a process X for those t and ω such that $t < \tau_n(\omega)$ — that is, those (t, ω) pairs to the left of the τ_n curve in this diagram — and then letting $n \to \infty$.

Proposition 23.1 (finite variation martingale) *If M is a continuous local martingale of locally finite variation then $M = M_0$ a.s.*

Proof: We can reduce this to the case where M is a true martingale with bounded variation and $M_0 = 0$ a.s. The reduction uses a localization argument: it suffices to show that $M^{\tau_n} = M_0$ a.s. for each n, and each M^{τ_n} is a true martingale. See the first paragraph of Kallenbergs proof (p. 330) for details, and Figure 23.1 for intuition.

Now for a fixed t and n, let $t_i = \frac{it}{n}$. Let:

$$\begin{aligned} \xi_n &\triangleq \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2 \\ &\leq \left(\max_i |M_{t_i} - M_{t_{i-1}}| \right) V_t \\ &\to 0 \text{ a.s.} \quad \text{since } V_t \text{ is bounded} \end{aligned}$$

Note that $\xi_n \leq V_t^2$ for all n and $EV_t^2 < \infty$, so the dominated convergence theorem implies $E\xi_n \to 0$ a.s. But:

$$\begin{aligned} E\xi_n &= E(M_t^2 - M_0^2) \quad \text{by orthogonality of martingale increments} \\ &= EM_t^2 \quad \text{because } EM_0^2 = 0 \end{aligned}$$

So $EM_t^2 = 0$, which implies $M_t = 0$ a.s.

We have proved this for arbitrary t, so we know $\forall t\ P(M_t = 0) = 1$. But we want to show $P(\forall t\ M_t = 0) = 1$. We do this by noting that $\forall t\ P(M_t = 0) = 1$ implies $P(\forall t \in \mathbb{Q}^+\ M_t = 0) = 1$, and then concluding that $P(\forall t\ M_t = 0) = 1$ because M has continuous paths. ■

This proposition is used in proving the uniqueness of the covariation process (Thm. 23.5).

23.2 Stochastic integral of a step function

We want to define the stochastic integral $\int_0^t Y dX$, where (X_t) and (Y_t) are both processes, and $\left(\int_0^t Y dX\right)$ is another process. Kallenberg also uses the notation $(Y \cdot X)_t$ as a synonym for $\int_0^t Y dX$. The following definition handles the easy special case where Y is a step process:

Definition 23.4 *Suppose we have a process X, stopping times $\tau_k \uparrow \infty$, random variables $\eta_k \in \mathcal{F}_{\tau_k}$, and a* **predictable step process***:*

$$V_t = \sum_k \eta_k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t)$$

That is, V_t equals η_1 on $(\tau_1, \tau_2]$, η_2 on $(\tau_2, \tau_3]$, etc. Then the **stochastic integral of the step process** *V with respect to X is:*

$$(V \cdot X)_t = \sum_k \eta_k (X^t_{\tau_{k+1}} - X^t_{\tau_k}) \tag{23.1}$$

Recall that $X^t_{\tau_k}$ is the process X stopped at time t, and evaluated at time τ_k. So if $\tau_k > t$, then $X^t_{\tau_k} = X_t$.

One way to understand the stochastic integral is to imagine that $X^t_{\tau_{k+1}} - X^t_{\tau_k}$ is the fluctuation of a market between times τ_k and τ_{k+1}, and η_k is our "bet, or the number of shares in the market that we own between τ_k and τ_{k+1}. Then $\int_0^t V dX$ is the amount we gain in the market up to time t. By extending our definition of the stochastic integral to handle processes other than step processes, we will be able to model investment strategies that change the bet continuously.

Note that since $(V \cdot X)$ depends only on the changes in X:

$$(V \cdot X)_t = (V \cdot (X - X_0))_t$$

Recall that a martingale M is in L^2 if $\sup_t EM_t^2 < \infty$.

Proposition 23.2 *For any continuous L^2-martingale M where $M_0 = 0$, and any predictable step process V where $|V| \le 1$, the process $(V \cdot M)$ is an L^2-martingale with $E(V \cdot M)^2_t \le EM^2_t$.*

Proof: First assume there are only a finite number of nonzero terms in V_t. We use the following lemma from Chapter 7 of Kallenberg:

Lemma 23.3 *If M is a continuous martingale, τ is a stopping time, and $\zeta \in \mathcal{F}_\tau$, then the process $(N_t) = (\zeta(M_t - M^\tau_t))$ is also a martingale.*

This process N_t is zero up to time τ, because for those times $M_t = M^\tau_t$. For $t \ge \tau$, $N_t = \zeta M_t - \zeta M_\tau$. We can rewrite the definition of $(V \cdot M)_t$ as a sum of processes of this form (see equation (1) in Chapter 17 of Kallenberg). So from the lemma and the assumption that the sum is finite, we can conclude that $(V \cdot M)_t$ is a martingale.

We still have to show $E(V \cdot M)^2_t \le EM^2_t$:

$$\begin{aligned} E(V \cdot M)^2_t &= E\left(\sum_k \eta_k (M^t_{\tau_{k+1}} - M^t_{\tau_k})\right)^2 \\ &= E\left(\sum_k \eta_k^2 (M^t_{\tau_{k+1}} - M^t_{\tau_k})^2\right) + 2E\left(\sum_{i<j} \eta_i \eta_j (M^t_{\tau_{j+1}} - M^t_{\tau_j})(M^t_{\tau_{i+1}} - M^t_{\tau_i})\right) \end{aligned}$$

But by the orthogonality of martingale increments, the second expectation is zero. So:

$$\begin{aligned}
E(V\cdot M)_t^2 &= E\left(\sum_k \eta_k^2 (M^t_{\tau_{k+1}} - M^t_{\tau_k})^2\right) \\
&\le E\left(\sum_k (M^t_{\tau_{k+1}} - M^t_{\tau_k})^2\right) \quad \text{since } |V|\le 1 \\
&= EM_t^2 \quad \text{by orthogonality of increments}
\end{aligned}$$

For the general case where V has infinitely many nonzero terms, take $V_j \to V$, where each V_j has finitely many nonzero terms. Then:

$$\begin{aligned}
E(V\cdot M)_t^2 &= E(\liminf_j (V_j\cdot M)_t)^2 \quad \text{by Fatous lemma} \\
&\le EM_t^2 \quad \text{by result proved above}
\end{aligned}$$

This proves the second claim in the lemma, but we still need to show $(V\cdot M)_t$ is a martingale. This is left as an exercise; the idea is to use dominated convergence. ■

23.3 The space $\mathcal{M}^2$

Definition 23.5 *For a fixed filtration $\mathcal{F}$, define the space:*

$$\mathcal{M}^2 = \{M : M \text{ is a continuous martingale with respect to } \mathcal{F} \text{ and is } L^2\text{-bounded}\}$$

It can be shown that if $M \in \mathcal{M}^2$, then there is some M_∞ such that $M_t \to M_\infty$ a.s. as $t \to \infty$. The proof of this result builds on the fact that $X_t \to X_\infty$ when X is a discrete L^2-martingale; we then consider countable subsequences of the indices t for (M_t). Furthermore, given an M_∞, we can recover the process (M_t) such that $M_t \to M_\infty$: by the definition of a continuous-time martingale, $M_t = E(M_\infty|\mathcal{F}_t)$ for each t.

Since each $M \in \mathcal{M}^2$ converges to some M_∞, we can define the norm:

$$\|M\| = \|M_\infty\|_2 = (EM_\infty^2)^{1/2}$$

Proposition 23.4 *For any $M \in \mathcal{M}^2$, let $M^* = \sup_t |M_t|$. Then $\|M^*\|_2 \le 2\|M\|$.*

Proof: Let $\bar{M}_t = \sup_{s\in[0,t]} |M_s|$. By the L^2 maximum inequality (Thm. 4.4.3 of Durrett), for any t:

$$\begin{aligned}
(E(\bar{M}_t)^2)^{1/2} &\le 2(EM_t^2)^{1/2} \\
&\le 2\sup_t (EM_t^2)^{1/2}
\end{aligned}$$

But $M_t = E(M_\infty|\mathcal{F}_t)$, so because conditioning reduces variance:

$$\begin{aligned}
(E(\bar{M}_t)^2)^{1/2} &\le 2\sup_t (EM_\infty^2)^{1/2} \\
&= 2(EM_\infty^2)^{1/2} = 2\|M\|
\end{aligned}$$

So $\|\bar{M}_t\|_2 \le 2\|M\|$ for each t, which implies $\|M^*\|_2 \le 2\|M\|$. ■

This proposition is used to prove that $\mathcal{M}^2$ is complete; see Lemma 17.4 in Kallenberg.

23.4 Covariation and quadratic variation

Theorem 23.5 *For any continuous local martingales M and N, there exists an a.s. unique continuous process $[M, N]$, called the* **covariation process** *of M and N, such that $[M, N]$ has locally finite variation, $[M, N]_0 = 0$, and $MN - [M, N]$ is a local martingale.*

The existence portion of the proof will be done in the next lecture. Assuming such an $[M, N]$ exists, its uniqueness follows from Prop. 23.1. Also, given uniqueness, it is obvious that the form $[M, N]$ must be symmetric and bilinear.

Definition 23.6 *If M is a continuous local martingale, the* **quadratic variation** *of M is $[M, M]$.*

Proposition 23.6 *For any continuous local martingales M and N and any stopping time τ:*

$$[M, N]^\tau = [M^\tau, N^\tau] = [M^\tau, N] \text{ a.s.}$$

Proof: The first inequality follows directly from the uniqueness of $[M, N]$. For the second, note that since $MN - [M, N]$ is a local martingale,

$$M^\tau N^\tau - [M, N]^\tau$$

is a local martingale. It can also be shown that whenever N is a local martingale,

$$M^\tau (N - N^\tau)$$

is a local martingale (Kallenberg cites his Corollary 7.14 for this fact). Adding the two local martingales together, we get another local martingale:

$$M^\tau N - [M, N]^\tau$$

But by Theorem 23.5, $M^\tau N - [M^\tau, N]$ is the unique local martingale obtained by subtracting a covariation process from $M^\tau N$, so it must be that $[M, N]^\tau = [M^\tau, N]$. ∎

So what have we done so far? Were trying to understand functions of martingales, and were starting with polynomials — specifically, what is the product of two martingales M and N? The product MN is not generally a martingale, but Theorem 23.5 says that $MN - [M, N]$ is a local martingale. As an example of this, consider $B_t^2 - [B_t] = B_t^2 - t$, which we already knew was a martingale.