Stat205A: Probability Theory (Fall 2002) **Lecture: 23-24**

Recurrence and Transience of Random Walks

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In this lecture, let $X_1, X_2, \cdots$ be i.i.d. and $S_n = X_1 + X_2 + \cdots + X_n,\ S_n = 0$. S_n is **a random walk**.

Theorem 23.1 *Let $X_1, X_2, \cdots$ be i.i.d. , $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$, τ is a stopping time. Conditional on $\tau < \infty$, $S_\tau, S_{\tau+1}, \cdots$ is a R.W.*[1] *started at S_τ. i.e. $X_{\tau+1}, X_{\tau+2}, \cdots$ are i.i.d. and independent of $\mathcal{F}_\tau$.*

Proof Sketch: Conditional on τ

$$
\begin{aligned}
& \mathbb{P}(\tau = n, (X_1, \cdots, X_\tau) \in A, (X_{\tau+1}, \cdots, X_{\tau+m}) \in B) \\
= \; & \mathbb{P}(\tau = n, (X_1, \cdots, X_n) \in A, (X_{n+1}, \cdots, X_{n+m} \in B) \\
= \; & \mathbb{P}(\tau = n, (X_1, \cdots, X_n) \in A)\mathbb{P}((X_1, \cdots, X_m) \in B)
\end{aligned}
$$

Summing over n gets the desired result. ∎

Definition 23.1 *The number $x \in \mathbb{R}$ is said to be* **a recurrent value** *for the R.W. S_n if for every $\epsilon > 0$, $\mathbb{P}(S_n \in (x \pm \epsilon)$*[2]*$) = 1$.*

Definition 23.2 *We say F is* **Lattice with period d** *if $F(\mathbb{Z}d) = 1$ and d is the greatest positive number with this property. Otherwise, there is no such d, and it's called* **Non-lattice**.

Example 23.1 $F = \frac{1}{2}(\delta_e + \delta_1)$ *is non-lattice.*

Theorem 23.2 *If R.W. S_n is lattice with range $\mathbb{Z}d$ as above, then either 1) or 2)*

1) each $x \in \mathbb{Z}d$ is recurrent,

2) each $x \in \mathbb{Z}d$ is transient.

Proof Sketch: Just Markov chain theory. ∎

Definition 23.3 *y is said to be* **possible** *if for every open interval I, there exists k, s.t. $\mathbb{P}(S_k \in I) > 0$.*

Lemma 23.3 *If x is recurrent and y is possible, then $x-y$ is recurrent.*

[1] random walk
[2] $x \pm \epsilon := [x - \epsilon, x + \epsilon]$

Proof Sketch: Take $\epsilon > 0$, then there exists k, s.t. $\mathbb{P}(|S_k - y| < \epsilon) > 0$. From **Theorem 23.1**

$$\begin{aligned} \mathbb{P}(|S_n - x| < 3\epsilon, f.o.^3) &\geq \mathbb{P}(|S_k - y| < \epsilon, |S_(k+n) - S_k - (x-y)| < 2\epsilon, f.o.) \\ &= \mathbb{P}(|S_k - y| < \epsilon)\, \mathbb{P}(S_n - (x-y)| < 2\epsilon, f.o.) \end{aligned}$$

If $\mathbb{P}(S_n - (x-y)| < 2\epsilon, f.o.) > 0$, then $\mathbb{P}(|S_n - x| < 3\epsilon, f.o.) > 0$, which is a contradiction! ∎

Theorem 23.4 *If R.W. S_n is non-lattice, then similarly either 1) or 2)*

1) each $x \in \mathbb{R}$ is recurrent,

2) each $x \in \mathbb{R}$ is transient.

Proof Sketch: Let G=$\{x \in \mathbb{R}$: x is recurrent$\}$. Suppose $G \neq \varnothing$, then

- It is clear that G^c is open, so G is closed[4].
- From the above lemma, if $x \in G$ and $y \in G$, then $x - y \in G$. Therefore, G is a group.

Since G is a closed subgroup of $\mathbb{R}$ and the R.W. is non-lattice, it follows that G=$\mathbb{R}$. ∎

Note: If $\mathbb{E}$(X) is defined , finite and not 0, then the R.W. is transient (i.e. {recurrent points}=$\varnothing$) by S.L.L.N[5].

Definition 23.4 $\mathbb{U}$: **potential measure**. *For any interval I,* $\mathbb{U}$ *(I):=*$\sum_n \mathbb{P}(S_n \in I) = \mathbb{E}(\sum_n \mathbf{1}_{(S_n \in I)})$.

Lemma 23.5 $\mathbb{P}(S_n \in (x \pm \epsilon/2)$ *for some n*$)\, \mathbb{U}(\pm\epsilon/2) \leq \mathbb{U}(x \pm \epsilon) \leq \mathbb{U}(\pm 2\epsilon)$

Proof Sketch: Let τ:=the first hit of $(x \pm \epsilon)$, then $S_{\tau+n} \in (x - \epsilon, x + \epsilon) \Rightarrow (S_{\tau+n} - S_\tau) \in (\pm 2\epsilon)$. Therefore from **Theorem 23.1**

$$\begin{aligned} \mathbb{U}(x \pm \epsilon) &= \mathbb{E}[\text{the number of times n that } S_n \in (x \pm \epsilon)] \\ &= \mathbb{E}[\text{the number of times n that} (S_{\tau+n} - S_\tau) \in (\pm\epsilon)] \\ &\leq \mathbb{E}[\text{the number of times n that } S_n \in (\pm 2\epsilon)] \\ &= \mathbb{U}(\pm 2\epsilon) \end{aligned}$$

Let τ_1:=the first hit of $(x \pm \epsilon/2)$. Use the same argument

$$\begin{aligned} \mathbb{P}(S_n \in (x \pm \epsilon/2) \text{ for some n})\, \mathbb{U}(\pm\epsilon/2) &= \mathbb{E}[\text{the number of times n that} (S_{\tau_1+n} - S_{\tau_1}) \in (\pm\epsilon/2)] \\ &\leq \mathbb{E}[\text{the number of times n that } S_n \in (x \pm \epsilon)] \\ &= \mathbb{U}(x \pm \epsilon) \end{aligned}$$

∎

Corollary 23.6 $\mathbb{U}(\pm k\epsilon) \leq (2k+1)\, \mathbb{U}(\pm\epsilon)$, $\forall\ k \in \mathbb{N}$.

[3]finitely often
[4]topologically closed
[5]Strong Law of Large Numbers

Proof Sketch: Cover $(-k\epsilon, k\epsilon)$ with (2k+1) intervals of the form $(x \pm \epsilon/2)$. Use the fact that $\mathbb{U}$ is a measure. ■

Proposition 23.7 *Either $\mathbb{U}(I) < \infty$ for all bounded intervals I (transient case) or $\mathbb{U}(x \pm \epsilon) = \infty$ for all possible x and all $\epsilon > 0$.*

Proof Sketch: Consider $\mathbb{U}(\pm\delta)$:

1) If $\mathbb{U}(\pm\delta) < \infty$ for some $\delta > 0$, then
$\mathbb{U}(\pm k\delta) \leq (2k+1)\mathbb{U}(\pm\delta) < \infty, \forall \ k \in \mathbb{N} \implies \mathbb{U}(I) < \infty$ for all bounded intervals I.

2) If $\mathbb{U}(\pm\delta) = \infty$ for all $\delta > 0$, then from **Lemma 23.5**
$\mathbb{P}(S_n \in (x \pm \delta/2)$ for some n) $\mathbb{U}(\pm\delta/2) \leq \mathbb{U}(x \pm \delta) \implies \mathbb{U}(x \pm \delta) = \infty$ for all $\delta > 0$ if x is possible.

■

Theorem 23.8 *Either $\mathbb{U}(\pm 1) < \infty$ and no x is recurrent or $\mathbb{U}(\pm 1) = \infty$ and every possible x is recurrent.*

Proof Sketch: If $\mathbb{U}(\pm 1) < \infty$, no x is recurrent by **Borel-Cantelli lemma**.
The other way:
If S_n is in an interval I only finitely often, consider τ:= the **last** time that $S_n \in I$.
Careful: τ is not a stopping time since $\{\tau = n\} = \{S_n \in I,\ S_{n+1} \in I,\ S_{n+2} \in I, \cdots\}$
$\{\tau = 0\} = \{S_n \in I,\ \text{for all } n\}, \quad \{\tau = \infty\} = \{S_n \in I,\ i.o.\}$.

Since $\mathbb{U}(\pm 1) = \infty$ by assumption, we know that $\mathbb{U}(\pm\epsilon) = \infty,\ \forall\ \epsilon > 0$ by estimate:
$\mathbb{U}(\pm k\epsilon) \leq (2k+1)\mathbb{U}(\pm\epsilon)\ for\ k \geq \frac{1}{\epsilon}$.
Let τ:=last time that the R.W. is in $(\pm\epsilon)$, then $\mathbb{P}(S_n \in (\pm\epsilon),\ f.o.) = \sum_n \mathbb{P}(\tau = n)$.

$$\begin{aligned}\{\tau = n\} &= \{S_n \in (\pm\epsilon)\} \cap \{S_{n+k} \notin (\pm\epsilon),\ \forall\ k \geq 1\} \\ &\supset \{S_n \in (\pm\epsilon)\} \cap \{S_{n+k} - S_n \notin (\pm 2\epsilon),\ \forall\ k \geq 1\}\end{aligned}$$

Therefore $\mathbb{P}(\tau = n) \geq \mathbb{P}(S_n \in (\pm\epsilon)\ \mathbb{P}(|S_k| \in 2\epsilon,\ \forall\ k \geq 1)$. Sum over n:

$$1 \geq \mathbb{P}(S_n \in \pm\epsilon,\ f.o.) \geq \mathbb{U}(\pm\epsilon)\,\mathbb{P}(|S_k| \geq 2\epsilon,\ \forall\ k \geq 1)$$

But $\mathbb{U}(\pm\epsilon) = \infty$, which forces the term $\mathbb{P}(|S_k| \geq 2\epsilon,\ \forall\ k \geq 1)$ to be 0.
Rewrite what we have proved:

$$\mathbb{U}(\pm 1) = \infty \implies \mathbb{P}(|S_n| \geq \delta,\ \forall\ n \geq 1) = 0 \text{ for all } \delta > 0$$

Finish the argument:

$$
\begin{aligned}
\mathbb{P}(S_n \in (\pm\epsilon),\ f.o.) &= \mathbb{P}(\tau < \infty \text{ and } S_\tau \in (\pm\epsilon)) \\
&= \lim_{k\to\infty} \mathbb{P}(\tau < \infty \text{ and } S_\tau \in (\pm\epsilon(1-\frac{1}{k}))) \\
&= \lim_{k\to\infty} \sum_{n=0}^{\infty} \mathbb{P}(S_n \in (\pm\epsilon(1-\frac{1}{k})) \text{ and } S_{n+j} \notin (\pm\epsilon),\ \forall\ j \geq 1) \\
&\leq \lim_{k\to\infty} \sum_{n=0}^{\infty} \mathbb{P}(S_n \in \pm\epsilon(1-\frac{1}{k}))\ \mathbb{P}(|S_j| \geq \frac{\epsilon}{k},\ \forall\ j \geq 1) \\
&= 0 \quad \textit{since } \mathbb{P}(|S_n| \geq \delta,\ \forall\ n \geq 1) = 0 \textit{ for all } \delta > 0.
\end{aligned}
$$

■

Key idea here: Think about the **last** time in the trip.

Theorem 23.9 (Chung-Fuchs Theorem) *Suppose* $\mathbb{E}|X_1| < \infty$*.*

- *If* $\mathbb{E}X_1 \neq 0$*, then the R.W. is transient(by S.L.L.N.)*
- *If* $\mathbb{E}X_1 = 0$*, then all possible points are recurrent.*

Proof Sketch: We'll show $\mathbb{U}(\pm 1) = \infty$ when $\mathbb{E}X_1 = 0$. We know

$$
\begin{aligned}
\mathbb{U}(\pm 1) &\geq (\frac{1}{2k+1})U(\pm k) \\
&= (\frac{1}{2k+1}) \sum_{n=0}^{\infty} \mathbb{P}(S_n \in (\pm k))
\end{aligned}
$$

Take $\epsilon > 0$, and choose k so that

$$\mathbb{P}(\tfrac{|S_n|}{n} < \epsilon) \geq \tfrac{1}{2} \text{ for all } n \geq k \text{ (by W.L.L.N. [6])}$$

For $k \leq n \leq \frac{k}{\epsilon}$, $\mathbb{P}(|S_n| < k) \geq \frac{1}{2}$. Hence

[6] Weak Law of Large Numbers

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{P}(S_n \in (\pm 1)) \quad &\geq \quad (\frac{1}{2k+1})\ \sum_{n=0}^{\infty}\mathbb{P}(S_n \in (\pm k)) \\
&\geq \quad (\frac{1}{2k+1})\ \sum_{k \leq n \leq \frac{k}{\epsilon}} \mathbb{P}(S_n \in (\pm k)) \\
&\geq \quad (\frac{1}{2k+1})\ \sum_{k \leq n \leq \frac{k}{\epsilon}} \frac{1}{2} \\
&\geq \quad \frac{1}{2}\ (\frac{1}{2k+1})\ (\frac{k}{\epsilon} - k) \\
&\geq \quad \frac{1}{2}\ (\frac{1}{3k})\ (\frac{k}{\epsilon} - k) \\
&\geq \quad \frac{1}{6}(\frac{1}{\epsilon} - 1) \\
&\rightarrow \quad \infty \quad as\ \epsilon \rightarrow 0.
\end{aligned}
$$

■