**Stat205A: Probability Theory (Fall 2002)** **Lecture: 26**

# Processes with Independent Increments

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## 26.1 Poisson Processes and Brownian Motion

Let $(F_t)_{t\geq 0}$ be a filtration. Usually, but not necessarily, $F_t = \sigma(X_s, 0 \leq s \leq t)$.

**Definition 26.1** *A real valued process $X_t$, $t \geq 0$, is an $F_t$-***Brownian Motion** *(BM) if*

*0) $X_t$ is $F_t$-measurable.*

*1) The mapping $t \to X_t(w)$ is continuous for almost every $w$.*

*2) For $s, t \geq 0$ the increment $X_{t+s} - X_s$ is normally distributed with mean 0 and variance $t$.*

*3) $X_{t+s} - X_s$ is independent of $F_s$.*

**Definition 26.2** *A real valued process $X_t$, $t \geq 0$, is an $F_t$-***Poisson Process** *with rate $\lambda$ or $PP(\lambda)$ if*

*0) $X_t$ is $F_t$-measurable.*

*1) The mapping $t \to X_t(w)$ is increasing, right continuous, and takes nonnegative integer values.*

*2) For $s, t \geq 0$ The increment, $X_{t+s} - X_s$, is a Poisson random variable with parameter $\lambda t$.*

*3) $X_{t+s} - X_s$ is independent of $F_s$.*

**Theorem 26.1** (Lévy) *A real-valued process $X_t, t \geq 0$ with $X_0 = 0$ and continuous paths is an $F_t$-Brownian Motion if and only if*

*1) $X_t$ is an $F_t$-martingale.*

*2) $X_t^2 - t$ is an $F_t$-martingale.*

**Theorem 26.2** (Watanabe) *A process $X_t, t \geq 0$ with $X_0 = 0$ and increasing, right continuous step function paths with all jumps of size 1 is an $F_t$-Poisson Process with rate $\lambda$ if and only if $X_t - \lambda t$ is an $F_t$-martingale.*

The proofs of the above two theorems require stochastic calculus and are not given here.

## 26.2 Construction of a Poisson Process

Let $T_0 = 0$ and for $r = 1, 2, \ldots$ let

$$T_r \ = \text{time of } r^{th} \text{ jump}$$

and let

$$N_t = \sum_{r=1}^{\infty} 1_{(T_r \le t)} \text{ and } W_r = T_r - T_{r-1}$$

.

**Theorem 26.3** *Assuming $(N_t)_{t\ge 0}$ is a simple counting process, it is a $PP(\lambda)$ process if and only if $W_1, W_2, W_3, \ldots$ are i.i.d. with*

$$P(W_r > t) = e^{-\lambda t}$$

.

**Proof:** See Durrett, Sec 2.6. ■

**Definition 26.3** *For any $r > 0$, a random variable, $\Gamma_r$, has* **gamma**$(r, \lambda)$ *distribution if*

$$P(\Gamma_r \in dt) = \frac{1}{\Gamma(r)} t^{r-1} \lambda^r e^{-\lambda t} dt$$

where $\Gamma(r)$ is a constant of normalization, called the *gamma function*.

**Claim 26.4** *Fore $r = 1, 2, \ldots$, the time $T_r$ of the $r$th point in a $PP(\lambda)$ has gamma$(r, \lambda)$ distribution.*

**Proof:** Using independence of increments of the Poisson process

$$\begin{aligned} P(T_r \in dt) &= P(r-1 \text{ arrivals in } (0,t) \text{ and } 1 \text{ arrival in } dt) \\ &= P(r-1 \text{ arrivals in } (0,t)) P(1 \text{ arrival in } dt) \\ &= e^{-\lambda t} \frac{(\lambda t)^{r-1}}{(r-1)!} \lambda dt \end{aligned}$$

■

Note that for each fixed $\lambda > 0$ the family of gamma$(r, \lambda)$ distributions forms a convolution semigroup, *i.e.*, if $F_r(t) = P(\Gamma_r \le t)$ is the *c.d.f.* of $\Gamma_r$ then

$$F_r * F_s = F_{r+s}$$

For $r, s = 0, 1, 2, \ldots$ this is obvious from the Poisson Process interpretation. That this is also true for all real $r, s > 0$ can be shown by computation.

## 26.3 Compound Poisson Process

Compound Poisson Process are frequently used to model losses in the insurance industry. Let $J_1, J_2, J_3, \ldots$ be *i.i.d.* random variables with some *c.d.f.* $F(x) = P(J_i \le x)$, and independent of a

Poisson process $(N_t)_{t\geq 0}$. The $J_i$ are now the jumps in our process. They could be interpreted, for example, as the losses associated with a sequence of automobile accidents.

Let

$$X_t = \sum_{i=1}^{N_t} J_i = \text{ loss accrued up to time t}$$

Notice that $(X_t)_{t\geq 0}$ has stationary independent increments. We can compute the characteristic function for $X_t$ as follows

$$\begin{aligned}
\phi_{X_t}(\theta) = E[e^{i\theta X_t}] &= E[e^{i\theta(J_1+J_2+\cdots+J_{N_t})}] \\
&= \sum_{n=0}^{\infty} P(N_t = n)E[e^{i\theta J_1}]^n \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}E[e^{i\theta J_1}]^n \\
&= e^{-\lambda t} exp(\lambda t E[e^{i\theta J_1}]) \\
&= exp(\lambda t E[e^{i\theta J_1} - 1])
\end{aligned}$$

For $t = 1$ and letting $F(dx) = P(J_1 \in dx)$ we have

$$\begin{aligned}
\phi_{X_1}(\theta) &= exp(\lambda \int (e^{i\theta x} - 1)F(dx)) \\
&= exp(\int (e^{i\theta x} - 1)L(dx))
\end{aligned}$$

where $L := \lambda F$ is a positive measure on $\mathbb{R}$ with total mass $\lambda$.

If for any Borel set $A$ we define

$$N(t, A) = \sum_{i=1}^{N_t} 1_{(J_i\in A)} = \text{ number of } J_i \text{ in A up to time t}$$

then $N(t, A)$ is Poisson random variable with parameter $\lambda t F(A)$. Also, $N(t, A)$ and $N(t, A^c)$ are independent. In fact if $A_1, A_2, A_3, \ldots, A_n$ are disjoint Borel subsets then $(N(t, A_i))_1^n$ are independent Poisson random variables with parameters $\lambda t F(A_i)$.