Stat205A: Probability Theory (Fall 2002) **Lecture: 12**

Characteristic Functions: Basic Ideas

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We'll begin with a question from last lecture: $X_n \xrightarrow{P} X$ implies $X_n \xrightarrow{d} X$

In fact, we can see that from the following inequalities:

$$P(X_n \le x) = P(X_n \le x, |X_n - X| \le \epsilon) + P(X_n \le x, |X_n - X| > \epsilon) \le P(X \le x + \epsilon) + P(|X_n - X| > \epsilon)$$
$$P(X \le x - \epsilon) = P(X \le x - \epsilon, |X_n - X| \le \epsilon) + P(X \le x - \epsilon, |X_n - X| > \epsilon) \le P(X_n \le x) + P(|X_n - X| > \epsilon)$$

An application of Helly's Selection Theorem

Theorem 12.1. Suppose we have a collection ζ of bounded continuous functions $f : \mathbb{R} \mapsto \mathbb{R}$, and this collection ζ is a determining class, i.e.

$$Ef(X) = Ef(Y) \ for \ \forall f \in \zeta \Rightarrow X \stackrel{d}{=} Y$$

and we have a sequence of r.v's X_n s.t.

- $Ef(X_n)$ has a limit as $n \to \infty \ \forall \ f \in \zeta$
- $(X_n)$is tight. (i.e. $lim_{x\to\infty} sup_n P(|X_n| > x) = 0$)

Then $\exists$ a distribution for r.v. X s.t. $X_n \xrightarrow{d} X$.

Proof. Helly and tightness tell us we can find a subsequence of $\{X_n\}$, which is $\{X_{n_k}\}$, s.t.$X_{n_k} \Rightarrow X$. Noticing the condition (1), we know

$$Ef(X_n) \to Ef(X) \ \ \forall f \in \zeta$$

If $X_n \not\Rightarrow X$, then there exists a subsequence of $\{X_n\}$,which is $\{X_{n_j}\}$,s.t.$X_{n_j} \Rightarrow Y$,where $Y \stackrel{d}{\neq} X$. (This statement will involve the use of Helly's Select Theorem and tightness again). However, from $X_{n_j} \Rightarrow Y$, we get

$$Ef(X_n) \to Ef(Y) \ \ \forall f \in \zeta$$

So

$$Ef(X) = Ef(Y) \ \ \forall f \in \zeta$$

Since ζ is a determining class, we get $X \stackrel{d}{=} Y$, then we have a contradiction! So the proof is finished. □

Important Method: Characteristic Functions

Definition:

$$\varphi_X(t) = Ee^{itX} = E\cos tX + iE\sin tX$$

Fact:
1) $|\varphi_X(t)| \le 1$

2) $t \mapsto \varphi_X(t)$*is continuous. (using DCT), but not necessarily differentiable. More generally,*
3) If $EX < \infty$*,then* φ_X *has n-continuous derivatives then*

$$\varphi_X(t) = 1 + itEX + \cdots + \frac{(it)^n}{n!} EX^n + o(t), ast \to 0$$

4) $\varphi_{-X}(t) = \varphi_X(-t) = \overline{\varphi_X(t)}$
5) Key fact (Uniqueness): If $\varphi_X(t) = \varphi_Y(t)$ *for all* $t \in \mathbb{R}$*,then* $X \stackrel{d}{=} Y$.
One proof is by using Fejer's theorem in Fourier Series. And we will show another approach by getting the inversion formula in next lecture.

Note: Mixture of c.f.'s are c.f.'s.
If we have n c.f.'s, $\varphi_1, \varphi_3, \cdots, \varphi_n$*, which are the c.f. of* $F_1, F_2, \cdots, F_n$ *respectively. Then* $\sum_{i=1}^{n} p_i \varphi_i(t)$ *is the c.f. of*

$$F(x) = \sum_{i=1}^{n} p_i F_i(x)$$

where $p_i \geq 0$ *and* $\sum_i = 0^n p_i = 1$.