Question 1 (of practice midterm): The Sprague-Grundy function of progressively bounded game is defined as follows: $g(t) = 0$, for any terminal position $t$, and

$$g(x) = \text{mex}\{g(y) : \exists \text{ a legal move } x \rightarrow y\}.$$

More precisely, $g$ is defined firstly on terminal positions, and then on those positions such that each legal move leads to a terminal one, and so on, iteratively. The fact that the game is progressively bounded implies that this procedure eventually does define $g$ on all moves.

For the case of the game in the question, note that:

- $g(0) = 0$
- $g(1) = 1$
- $g(1, 1) = \text{mex}\{g(1)\} = 0$
- $g(2) = \text{mex}\{g(1, 1), g(1, g(0))\} = 2$
- $g(1, 1, 1) = \text{mex}\{g(1, 1)\} = 1$
- $g(2, 1) = \text{mex}\{g(1, 1, 1), g(1, 1), g(2)\} = 3$,

so that $g(3) = \text{mex}\{g(2, 1), g(2), g(1), g(0)\} = 4$.

We may show that $g(1000) = 999$ by proving by induction on $k$ that

$$g(4k) = 4k-1, \ g(4k+1) = 4k+1, \ g(4k+2) = 4k+2, \ g(4k+3) = 4k+4. \ (1)$$

Indeed, suppose that this holds for $k \leq k_0 - 1$. Note that

$$g(4k_0) = \text{mex}\{g(i) : i < 4k_0, \ g(i, 4k_0 - i) : 1 \leq i \leq 4k_0 - 1\}.$$

Note that $4k_0 - 1 = \text{mex}\{g(i) : i < 4k_0\}$ by the inductive hypothesis. We thus have to show that $g(i, 4k_0 - i) \neq 4k_0 - 1$ for $i \in \{1, \ldots, 4k_0 - 1\}$. This follows from the direct sum formula $g(i, 4k_0 - i) = g(i) \oplus g(4k_0 - i)$ and the inductive hypothesis. We might try to prove the other cases in (1) by a similar argument.

Given a position of pile sizes 13, 24 and 17, we should play a move so that the nim sum of the Sprague-Grundy function on the piles is zero. An example of such a move is $13 \rightarrow 6$. 

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**Question 2:** In the payoff matrix, the last column dominates the first. In the new matrix, $3/4$ last column + $1/4$ first column is dominated by the middle column. So we have reduced to

\[
\begin{array}{c|cc}
II & 0 & 7 \\
I & 6 & 1 \\
\end{array}
\]

The solution of this game: player II plays the first column with a probability $1/2$, while player I plays the first row with probability $5/12$. The value of the game is $7/2$.

In the original game, player I has optimal strategy $(5/12, 7/12)$, and player II has optimal strategy $(0, 1/2, 0, 1/2)$.

**Question 3:** The matrix has the form

\[
\begin{array}{c|cc}
II & 1/2 & 0 & 0 \\
I & 1 & 1/2 & 0 \\
& 0 & 1/2 & 1 \\
& 0 & 0 & 1/2 \\
\end{array}
\]

By invariance, we obtain,

\[
\begin{array}{c|cc}
II & 1/4 & 1/2 \\
I & 1/4 & 3/4 \\
\end{array}
\]

where the first column or row corresponds to the first or fourth in the original game, and the second, to the second or third. If player II plays the first column in the reduced game with probability $p_1$, then

\[
1/2p_1 + 1/4(1 - p_1) = 1/4p_1 + 3/4(1 - p_1).
\]

That is, $p_1 = 2/3$. By symmetry, player I plays the first row with probability $2/3$. The value of the game is $5/12$. An optimal strategy in the original game for either the crocodile or the zebra is given by the probability vector

\[(1/3, 1/6, 1/6, 1/3)\].

**Question 1 (separate homework):** The payoff matrix is given by:
The entry whose coordinates are $C$ and $LR$ is a maximum in its column and a minimum in its row. This means that game may be solved by deterministic strategies. Player $I$ puts the restaurant at Central and player $II$ puts one at Left and one at Right. The value of the game to Company $I$ is $1/4$. 

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